GJMS operator on Riemannian manifolds

TAHRI Kamel

Abou Bekr Belkaid Univ, Tlemcen
Algeria

Tuniso-Libanese Workshop, 30-31 Oct 2019, Tunisia

Outlines

Outline

Outlines

Outline

Outline

1. **Introduction and Motivation**
    1. **Conformal Laplacian**

Outline

Outline

Outline

1. **Introduction and Motivation**
   1. **Conformal Laplacian**
   2. **Paneitz-Branson Operator**
   3. **GJMS operator**
2. **Position of Problem**

Outlines

Outline

Introduction and Motivation

- **Introduction and Motivation**

Introduction and Motivation

- **Introduction and Motivation**

1. **Conformal Laplacian**

- **Introduction and Motivation**

1. **Conformal Laplacian**

- **Def:** The most well-known example of second order conformally covariant operator is the "**Conformal Laplacian"**:

$$g \longmapsto L_g^n := \Delta_g + \frac{n-2}{4\,(n-1)} S_g$$

and its associated curvature is precisely the scalar curvature S_g.

- **Introduction and Motivation**

1. **Conformal Laplacian**

- **Def:** The most well-known example of second order conformally covariant operator is the "**Conformal Laplacian**":

$$g \longmapsto L_g^n := \Delta_g + \frac{n-2}{4\,(n-1)} S_g$$

and its associated curvature is precisely the scalar curvature S_g.

- The conformal transformation law is usually regarded as: for a change of metric $\tilde{g} = u^{\frac{4}{n-2}} \times g$, $u > 0$ where $2^* = \frac{2n}{n-2}$

- **Introduction and Motivation**

1. **Conformal Laplacian**

- **Def:** The most well-known example of second order conformally covariant operator is the "**Conformal Laplacian"**:

$$g \longmapsto L_g^n := \Delta_g + \frac{n-2}{4\left(n-1\right)} S_g$$

and its associated curvature is precisely the scalar curvature S_g.

- The conformal transformation law is usually regarded as: for a change of metric $\tilde{g} = u^{\frac{4}{n-2}} \times g$, $u > 0$ where $2^* = \frac{2n}{n-2}$
  1. Then,

$$\forall \varphi \in C^\infty(M) : L_g^n\left(u.\varphi\right) = u^{2^*-1} \times L_{\tilde{g}}^n\left(\varphi\right)$$

- **Introduction and Motivation**

1. **Conformal Laplacian**

- **Def:** The most well-known example of second order conformally covariant operator is the "**Conformal Laplacian"**:

$$g \longmapsto L_g^n := \Delta_g + \frac{n-2}{4(n-1)} S_g$$

and its associated curvature is precisely the scalar curvature S_g.

- The conformal transformation law is usually regarded as: for a change of metric $\tilde{g} = u^{\frac{4}{n-2}} \times g$, $u > 0$ where $2^* = \frac{2n}{n-2}$
  1. Then,
  $$\forall \varphi \in C^\infty(M) : L_g^n (u.\varphi) = u^{2^*-1} \times L_{\tilde{g}}^n (\varphi)$$
  2. If $\varphi = 1$
  $$\Delta_g u + \frac{(n-2) S_g}{4(n-1)} u = \frac{(n-2)}{4(n-1)} S_{\tilde{g}} u^{2^*-1}$$

Introduction and Motivation

- **Introduction and Motivation**

[2]

Introduction and Motivation

- **Introduction and Motivation**

  [2]. **Yamabe Problem:**

Introduction and Motivation

- **Introduction and Motivation**

  [2]. **Yamabe Problem:**

- **Def:** Given (M^n, g) compact riemannian manifold without boundary, find $\tilde{g} \in [g]$ such that $S_{\tilde{g}} = \lambda$.

Introduction and Motivation

- **Introduction and Motivation**

  [2]. **Yamabe Problem:**

- **Def:** Given (M^n, g) compact riemannian manifold without boundary, find $\tilde{g} \in [g]$ such that $S_{\tilde{g}} = \lambda$.

1. Equivalent to finding solution of

$$\Delta_g u + \frac{(n-2)\, S_g}{4\,(n-1)} u = \frac{(n-2)}{4\,(n-1)} \lambda u^{2^*-1} \tag{1}$$

Introduction and Motivation

- **Introduction and Motivation**

  [2]. **Yamabe Problem:**

- **Def:** Given (M^n, g) compact riemannian manifold without boundary, find $\tilde{g} \in [g]$ such that $S_{\tilde{g}} = \lambda$.

1. Equivalent to finding solution of

$$\Delta_g u + \frac{(n-2)\, S_g}{4\,(n-1)} u = \frac{(n-2)}{4\,(n-1)} \lambda u^{2^*-1} \quad (1)$$

2. Standard Calculs of variation techniques fail to apply in !!! "**Critical Exponent of Sobolev"** !!!!!.

- **Introduction and Motivation**

[3]

- **Introduction and Motivation**

  [3]. **Paneitz-Branson Operator:**

- **Introduction and Motivation**

  [3]. **Paneitz-Branson Operator:**

- **Def:** When $n \geq 5$, $Q - curvature$ and the Paneitz-Branson P_g^n are higher order analogue of scalar curvature and the Conformal Laplacian respectively:

$$P_g^n(u) := \Delta_g^2(u) + \mathrm{div}_g\left(\left(a_n S_g.g - b_n Ric_g\right).du\right) + c_n Q_g^n u$$

and

$$Q_g^n := \frac{\Delta_g S_g}{2(n-1)} + \frac{n^3 - 4n^2 + 16n - 16}{8\left(n-1\right)^2\left(n-2\right)^2} S_g^2 - \frac{2}{\left(n-2\right)^2}\left|Ric_g\right|_g^2$$

- **Introduction and Motivation**

  [3]. **Paneitz-Branson Operator:**

- **Def:** When $n \geq 5$, $Q - curvature$ and the Paneitz-Branson P_g^n are higher order analogue of scalar curvature and the Conformal Laplacian respectively:

$$P_g^n(u) := \Delta_g^2(u) + \mathrm{div}_g\left(\left(a_n S_g.g - b_n Ric_g\right).du\right) + c_n Q_g^n u$$

and

$$Q_g^n := \frac{\Delta_g S_g}{2(n-1)} + \frac{n^3 - 4n^2 + 16n - 16}{8\left(n-1\right)^2\left(n-2\right)^2} S_g^2 - \frac{2}{\left(n-2\right)^2}\left|Ric_g\right|_g^2$$

- such that:

$$a_n := \frac{\left(n-2\right)^2 + 4}{2\left(n-1\right)\left(n-2\right)}, b_n := \frac{4}{\left(n-2\right)}, c_n := \frac{n-4}{2}$$

and Ric_g is the Ricci curvature.

Introduction and Motivation

- **Introduction and Motivation**

  [4]

Introduction and Motivation

- **Introduction and Motivation**

  [4]. **Conformal Covariant Property:**

Introduction and Motivation

- **Introduction and Motivation**

  [4]. **Conformal Covariant Property:**

- If $\tilde{g} = u^{\frac{4}{n-4}} g$, $u > 0$, we have:

$$\forall \varphi \in C^{\infty}(M) : P_g^n(\varphi u) = u^{\frac{n+4}{n-4}} P_{\tilde{g}}^n(\varphi) \text{ "Conformal Covariant"}$$

Introduction and Motivation

- **Introduction and Motivation**

  [4]. **Conformal Covariant Property:**

- If $\tilde{g} = u^{\frac{4}{n-4}} g$, $u > 0$, we have:

$$\forall \varphi \in C^{\infty}(M) : P_g^n(\varphi u) = u^{\frac{n+4}{n-4}} P_{\tilde{g}}^n(\varphi) \text{ "Conformal Covariant"}$$

- If $\varphi = 1$, then

$$P_g^n(u) = \frac{n-4}{2} Q_{\tilde{g}}^n u^{\frac{n+4}{n-4}}$$

Introduction and Motivation

- **Introduction and Motivation**

  [5]

Introduction and Motivation

- **Introduction and Motivation**

  [5].**Prescribed Q-cuvature:**

Introduction and Motivation

- **Introduction and Motivation**

  [5].**Prescribed Q-cuvature:**

- **Def:** Given $f \in C^{\infty}(M)$, find $\tilde{g} \in [g]$ such that $f(x) = Q^{n}_{\tilde{g}}$.

Introduction and Motivation

- **Introduction and Motivation**

  [5].**Prescribed Q-cuvature:**

5. **Def:** Given $f \in C^{\infty}(M)$, find $\tilde{g} \in [g]$ such that $f(x) = Q^n_{\tilde{g}}$.

1. Equivalent to finding solution of

$$P^n_g(u) = f(x)\, u^{\frac{n+4}{n-4}}. \tag{2}$$

Introduction and Motivation

- **Introduction and Motivation**

  [5].**Prescribed Q-cuvature:**

5 **Def:** Given $f \in C^{\infty}(M)$, find $\tilde{g} \in [g]$ such that $f(x) = Q^{n}_{\tilde{g}}$.

1. Equivalent to finding solution of

$$P^{n}_{g}(u) = f(x)\, u^{\frac{n+4}{n-4}}. \tag{2}$$

2. Main Difficulties:

Introduction and Motivation

- **Introduction and Motivation**

  [5].**Prescribed Q-cuvature:**

5 **Def:** Given $f \in C^{\infty}(M)$, find $\tilde{g} \in [g]$ such that $f(x) = Q^{n}_{\tilde{g}}$.

1. Equivalent to finding solution of

$$P^{n}_{g}(u) = f(x)\, u^{\frac{n+4}{n-4}}. \tag{2}$$

2. Main Difficulties:
   1. Standard Calculs of variation techniques fail to apply in !!! "**Critical Exponent of Sobolev"** !!!!!.

Introduction and Motivation

- **Introduction and Motivation**

  [5].**Prescribed Q-cuvature:**

5 **Def:** Given $f \in C^{\infty}(M)$, find $\tilde{g} \in [g]$ such that $f(x) = Q^n_{\tilde{g}}$.

1. Equivalent to finding solution of

$$P^n_g(u) = f(x)\, u^{\frac{n+4}{n-4}}. \qquad (2)$$

2. Main Difficulties:
   1. Standard Calculs of variation techniques fail to apply in !!! "**Critical Exponent of Sobolev"** !!!!!.
   2. !!! **Lack of a Maximum Principle** $\Longrightarrow$ **Solution u is not necessary Positive** !!!

Introduction and Motivation

- **Introduction and Motivation**

  [7]

Introduction and Motivation

- **Introduction and Motivation**

  [7]. **GJMS Operator:**

- **Introduction and Motivation**

  [7]. **GJMS Operator:**

- **Def:** Given (M^n, g) compact riemannian manifold.The **GJMS** operator is defined as:

$$P_g^k : C^\infty(M) \to C^\infty(M)$$

by

$$P_g^k(u) := \Delta_g^k(u) + \sum_{l=0}^{l=k-1} (-1)^l \nabla_g^{j_l \ldots j_1} \left(A_{l_{i_l \ldots i_1, j_l \ldots j_1}} \nabla_g^{i_l \ldots i_i} u \right)$$

where A_l is a smooth symetric T_{2l}^0-tensor field on M and $\Delta_g := -\operatorname{div}_g(\nabla_g)$ is the Laplacian Beltrami operator.

Introduction and Motivation

- **Introduction and Motivation**

  [8]

Introduction and Motivation

- **Introduction and Motivation**

  [8]. **The Behavior of GJMS Operator:**

Introduction and Motivation

- **Introduction and Motivation**

  [8]. **The Behavior of GJMS Operator:**

- One of the pertinent geometric behavior of P_g^k which is conformally convariant in the sense that: for all $\varphi \in C^\infty(M)$, $\varphi > 0$ and $\tilde{g} := \varphi^{\frac{4}{n-2k}} g$:

$$P_g^k(\varphi u) = \varphi^{\frac{n+2k}{n-2k}} . P_{\tilde{g}}^k(u)$$

Introduction and Motivation

- **Introduction and Motivation**

  [8]. **The Behavior of GJMS Operator:**

- One of the pertinent geometric behavior of P_g^k which is conformally convariant in the sense that: for all $\varphi \in C^\infty(M)$, $\varphi > 0$ and $\tilde{g} := \varphi^{\frac{4}{n-2k}} g$:

$$P_g^k(\varphi u) = \varphi^{\frac{n+2k}{n-2k}} . P_{\tilde{g}}^k(u)$$

- Therefore, this geometric quantity has been obtained when we take $\varphi \equiv 1$ as

$$Q_g^k := \frac{2}{n-2k} P_g^k(1)$$

Introduction and Motivation

- **Introduction and Motivation**

  [9]

Introduction and Motivation

- **Introduction and Motivation**

  [9]. **Prescribed Q-curvature for GJMS Operator:**

- **Introduction and Motivation**

  [9]. **Prescribed Q-curvature for GJMS Operator:**

- **Pb:** Mazumdar has proved in [1] the existence of $u \in C^{\infty}(M), u > 0$ and $f \in C^{\infty}(M)$ given and he established the following results of the following equation:

$$P_g^k(u) = f(x).\, |u|^{2_k^{\sharp}-2}\, u \qquad \text{in} \qquad M \tag{3}$$

Introduction and Motivation

- **Introduction and Motivation**

  [9]. **Prescribed Q-curvature for GJMS Operator:**

- **Pb:** Mazumdar has proved in [1] the existence of $u \in C^{\infty}(M), u > 0$ and $f \in C^{\infty}(M)$ given and he established the following results of the following equation:

$$P_g^k(u) = f(x).\,|u|^{2_k^{\sharp}-2}\,u \qquad \text{in} \qquad M \tag{3}$$

- **Sobolev Space** $H_k^2(M)$:

$$H_k^2(M) := \left\{ u \in C^{\infty}(M) : \sum_{m=0}^{m=k} \left\| \nabla_g^m u \right\|_{L^2(M)} < +\infty \right\}.$$

Introduction and Motivation

- **Introduction and Motivation**

  [9]

Introduction and Motivation

- **Introduction and Motivation**

  [9]. **Prescribed Q-curvature for GJMS Operator:**

- **Introduction and Motivation**

  [9]. **Prescribed Q-curvature for GJMS Operator:**

- **Theorem:** Let (M, g) be compact Riemannian manifold of dimension $n > 2k$ without boundary with $k \geq 1$. Let $f \in C^{0,\alpha}(M)$ positive function. We assume that P_g^k is coercive on $H_k^2(M)$.

- **Introduction and Motivation**

  [9]. **Prescribed Q-curvature for GJMS Operator:**

- **Theorem:** Let (M, g) be compact Riemannian manifold of dimension $n > 2k$ without boundary with $k \geq 1$. Let $f \in C^{0,\alpha}(M)$ positive function. We assume that P_g^k is coercive on $H_k^2(M)$.

  1. Suppose that

$$\inf_{u \in S_f} \int_M P_g^k(u).u d\mu_g < \frac{1}{K(n,k).\left(\max_{x \in M} f(x)\right)^{\frac{2}{2_k^\sharp}}}$$

where

$$S_f := \left\{ u \in H_k^2(M) : \int_M f(x).\left|u\right|^{2_k^\sharp} d\mu_g = 1 \right\}$$

and

$$K(n,k) := \inf_{u \in D^{k,2}(\mathbb{R}^n)-\{0\}} \frac{\int_{\mathbb{R}^n} \left(\Delta^{\frac{k}{2}} u\right)^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2_k^\sharp} dx\right)^{\frac{2}{2_k^\sharp}}}$$

- **Introduction and Motivation**

  [9]. **Prescribed Q-curvature for GJMS Operator:**

- **Theorem:** Let (M, g) be compact Riemannian manifold of dimension $n > 2k$ without boundary with $k \geq 1$. Let $f \in C^{0,\alpha}(M)$ positive function. We assume that P_g^k is coercive on $H_k^2(M)$.

  1. Suppose that

$$\inf_{u \in S_f} \int_M P_g^k(u).ud\mu_g < \frac{1}{K(n,k).\left(\max_{x \in M} f(x)\right)^{\frac{2}{2_k^\sharp}}}$$

  where

$$S_f := \left\{ u \in H_k^2(M) : \int_M f(x).\left|u\right|^{2_k^\sharp} d\mu_g = 1 \right\}$$

  and

$$K(n,k) := \inf_{u \in D^{k,2}(\mathbb{R}^n) - \{0\}} \frac{\int_{\mathbb{R}^n} \left(\Delta^{\frac{k}{2}} u\right)^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2_k^\sharp} dx\right)^{\frac{2}{2_k^\sharp}}}$$

  2. Then there exists a solution $u \in C^{2k}(M)$ to the equation (3)

Position of Problem

- **Position of Problem**

  [10]

Position of Problem

- **Position of Problem**

  [10] **Main Problem:**

Position of Problem

- **Position of Problem**

  [10] **Main Problem:**

10 **Pb:** In this presentation, we consider the multiplicity results of solutions of the following nonhomogenous $2k^{th}$ order elliptic equation involving GJMS's operator:

$$P_g^k(u) = f(x)\,|u|^{2_k^\sharp - 2}\,u + h(x) \tag{4}$$

Where f is a C^∞-function on M with $f > 0$ and h belongs to $\left(H_k^2(M)\right)^*$ and also, $2_k^\sharp = \frac{2n}{n-2k}$ is the critical Sobolev's exponent for the embedding $H_k^2(M) \subset L^{2_k^\sharp}(M)$. Now, we define when P_g^k is coercive our working norm as follow: for all $u \in H_k^2(M)$:

$$\|u\|^2_{P_g^k} := \int_M P_g^k(u).u d\mu_g$$

Position of Problem

- . **Position of Problem**

  [11]

Position of Problem

- . **Position of Problem**

  [11]**Technical Methods:**

Position of Problem

- . **Position of Problem**

  [11]**Technical Methods:**

11 Throughout this section, we consider the energy functional J, for each $u \in H_k^2(M)$,

$$J(u) = \frac{1}{2}\|u\|^2_{P_g^k} - \int_M h(x).ud\mu_g - \frac{1}{2_k^\sharp}\int_M f(x).|u|^{2_k^\sharp}d\mu_g$$

Position of Problem

- . **Position of Problem**

  [11]**Technical Methods:**

11 Throughout this section, we consider the energy functional J, for each $u \in H_k^2(M)$,

$$J(u) = \frac{1}{2}\|u\|^2_{P_g^k} - \int_M h(x).ud\mu_g - \frac{1}{2_k^\sharp}\int_M f(x).|u|^{2_k^\sharp} d\mu_g$$

- Define:

$$\alpha := \min_{u \in N} J(u)$$

- . **Position of Problem**

  [11]**Technical Methods:**

11 Throughout this section, we consider the energy functional J, for each $u \in H_k^2(M)$,

$$J(u) = \frac{1}{2}\|u\|^2_{P_g^k} - \int_M h(x).ud\mu_g - \frac{1}{2_k^\sharp}\int_M f(x).|u|^{2_k^\sharp} d\mu_g$$

- Define:

$$\alpha := \min_{u \in N} J(u)$$

- Put

$$\Phi(u) = \langle \nabla J(u),\ u\rangle = \|u\|^2_{P_g^k} - \int_M h(x).ud\mu_g - \frac{1}{2_k^\sharp}\int_M f(x).|u|^{2_k^\sharp} d\mu_g$$

Position of Problem

- . **Position of Problem**

  [11]**Technical Methods:**

11 Throughout this section, we consider the energy functional J, for each $u \in H_k^2(M)$,

$$J(u) = \frac{1}{2}\|u\|^2_{P_g^k} - \int_M h(x).u d\mu_g - \frac{1}{2_k^\sharp}\int_M f(x).|u|^{2_k^\sharp} d\mu_g$$

- Define:

$$\alpha := \min_{u \in N} J(u)$$

- Put

$$\Phi(u) = \langle \nabla J(u),\ u \rangle = \|u\|^2_{P_g^k} - \int_M h(x).u d\mu_g - \frac{1}{2_k^\sharp}\int_M f(x).|u|^{2_k^\sharp} d\mu_g$$

- Define,

$$N = \{u \in N :\ \Phi(u) = 0\}$$

- **Existence Result of Solution**

  [12]

- **Existence Result of Solution**

  [12]. **Key ingredients of the Theorem's Proof:**

- **Existence Result of Solution**

  [12]. **Key ingredients of the Theorem's Proof:**

12 - **Lemma:** Let (M, g) be a Riemannian compact smooth n-manifold, $n \geq 5$. Let f is a C^{∞}-function on M with $f > 0$ and $h \in \left(H_k^2(M)\right)^*$ such that $h \neq 0$ satisfies the condition (3). Then,:
  (i).If u_o is a local minimizer J on N ,then

$$\nabla J(u_o) = 0 \quad \text{in} \quad \left(H_k^2(M)\right)^*$$

  (ii). The functional J is bounded from below and coercive on N.

Multiplicity Results

- **Notation:**

  [13]

Multiplicity Results

- **Notation:**

  [13]

- We split N as follow:

$$N^{+} = \{u \in N : \ \langle \nabla \Phi(u), \ u \rangle > 0\}$$

$$N^{-} = \{u \in N : \ \langle \nabla \Phi(u), \ u \rangle < 0\}$$

Multiplicity Results

- **Notation:**

  [13]

- We split N as follow:

$$N^{+} = \{u \in N : \ \langle \nabla \Phi(u), \ u \rangle > 0\}$$

$$N^{-} = \{u \in N : \ \langle \nabla \Phi(u), \ u \rangle < 0\}$$

- and

$$N^{0} = \{u \in N : \ \langle \nabla \Phi(u), \ u \rangle = 0\}$$

Multiplicity Results

- **Multiplicity Results**

  [13]

Multiplicity Results

- **Multiplicity Results**

  [13]**Key ingredients of the Theorem's Proof:**

Multiplicity Results

- **Multiplicity Results**

  [13]**Key ingredients of the Theorem's Proof:**

13 **Lemma:** Let (M, g) be a Riemannian compact smooth n-manifold, $n \geq 5$. Under the previous assumption (3), then we have:

Multiplicity Results

- **Multiplicity Results**

  [13]**Key ingredients of the Theorem's Proof:**

13 **Lemma:** Let (M, g) be a Riemannian compact smooth n-manifold, $n \geq 5$. Under the previous assumption (3), then we have:

- For all $u \in N^-$:

$$\alpha^- := \inf_{u \in N^-} J(u) < 0.$$

Multiplicity Results

- **Multiplicity Results**

  [13]**Key ingredients of the Theorem's Proof:**

13 **Lemma:** Let (M, g) be a Riemannian compact smooth n-manifold, $n \geq 5$. Under the previous assumption (3), then we have:

- For all $u \in N^-$:

$$\alpha^- := \inf_{u \in N^-} J(u) < 0.$$

- For all $u \in N^+$:

$$\alpha^+ := \inf_{u \in N^+} J(u) > 0.$$

[13]**Main Result and Generic Theorem:**

The following theorem is our main result.

[13]**Main Result and Generic Theorem:**

The following theorem is our main result.

13 **Theorem:** Let (M, g) be compact Riemannian manifold of dimension $n > 2k$ without boundary with $k \geq 1$. Let $f \in C^{\infty}(M)$ positive function and $h \in \left(H_k^2(M)\right)^*$ such that $h \neq 0$ satisfies the condition (3). We assume that P_g^k is coercive on $H_k^2(M)$. Suppose that

$$\sup_{t \geq 0} J\left(u_o + tu_\epsilon\right) < c_o + \frac{k}{n\left[K\left(n, k\right)\right]^{\frac{n}{2k}}\left[\max_{x \in M} f\left(x\right)\right]^{\frac{n-2k}{2k}}}$$

and at a point a where f atteints its maximum the following condition

$$\frac{\Delta f\left(a\right)}{f\left(a\right)} < \frac{2\left(k+4\right)n^2 - 8\left(k+1\right)n - 24k}{3\left(n-2k\right)\left(n+2\right)(n-6)} S_g\left(a\right)$$

holds. Then, the equation (2) has two non trivial solutions.

Bibliography

1 Bibliography

- Ambrosetti A, Rabinowitz P, Dual Variational Methods in Critical Point Theory and Applications, J. Functional Analysis, Vol 11, (1973), 349-381.
- Brézis H, Lieb E, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1983), 486-490.
- Charles Fefferman and C. Robin Graham, Conformal invariants, Astérisque Numero Hors Serie (1985), 95–116. The mathematical heritage of ´Elie Cartan (Lyon, 1984).
- Ekeland I, On the variational principle, J. Math. Anal. Appl, 47 (1974), 324-353.
- Matthew Gursky and Andrea Malchiodi, A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature, J. Eur. Math. Soc. (JEMS) 17 (2015), no. 9, 2137–2173.

Robin C. Graham, Ralph Jenne, Lionel J. Mason, and George A. J. Sparling, Conformally invariant powers of the Laplacian. I. Existence, J. London Math. Soc. (2) 46 (1992), no. 3, 557–565.

Saikat Mazumdar, GJMS-type Operators on a compact Riemannian manifold: Best constants and Coron-type solutions, arXiv.org: math: arXiv:1512.02126 (2015).

Stephen M. Paneitz, A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds, SIGMA 4 (2008), Paper 036, 3.

Tahri K, On GJMS-operator with Concave-convex Nonlineaaities Involving Critical Sobolev Exponent (Preprint).

Tahri K, Existence and Multiplicity Results of GJMS-operator (Preprint).

Tarantello G, On nonhomogeneous elliptic equations involving critical Sobolev exponent, Ann. Inst. H. Poincaré Anal. Non. Linéaire 9 (1992), 281-304.