Path with Two Blocks
(A New Elementary Proof of El Sahili Conjecture)

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Definitions

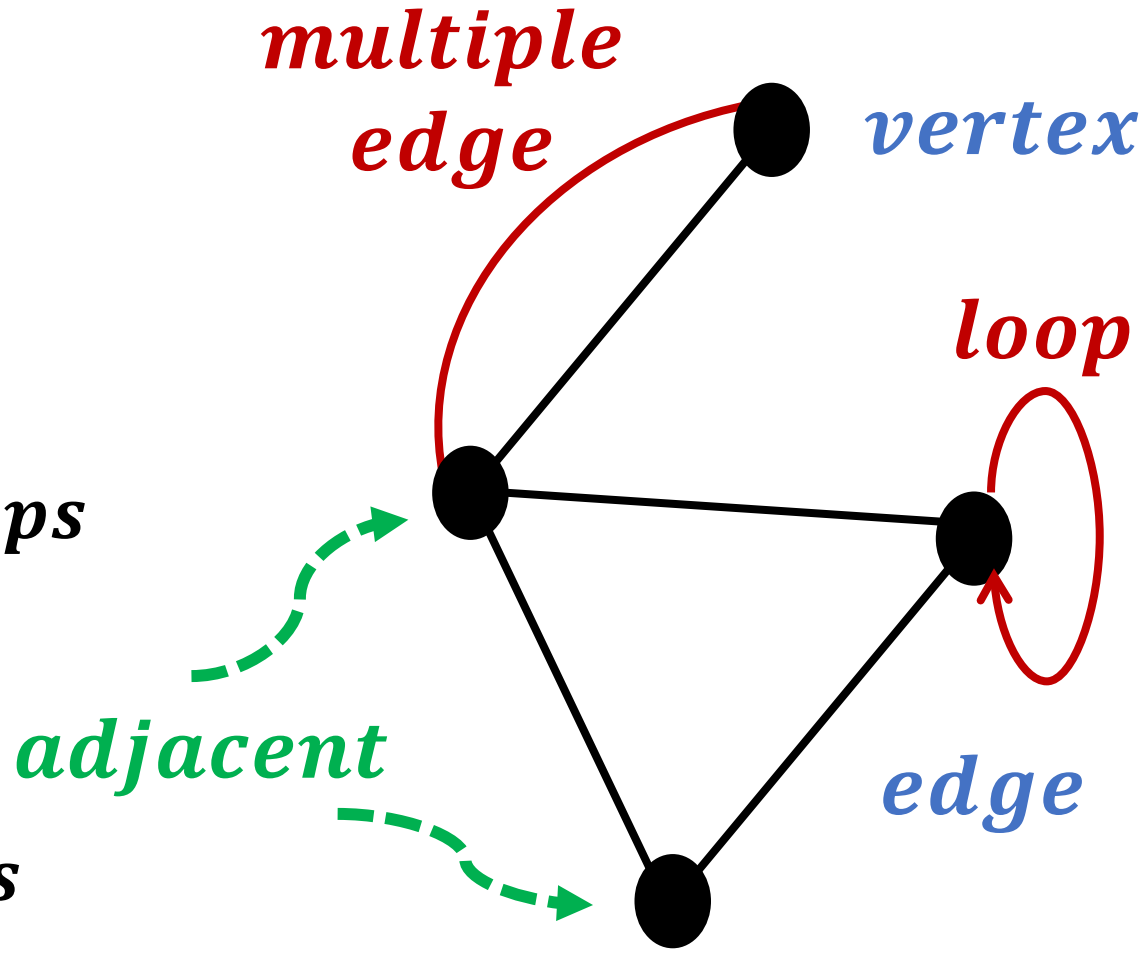
$V(G)$: *set of vertices*

$E(G)$: *set of edges*

Simple Graph:
a graph containing neither loops nor multiple edges

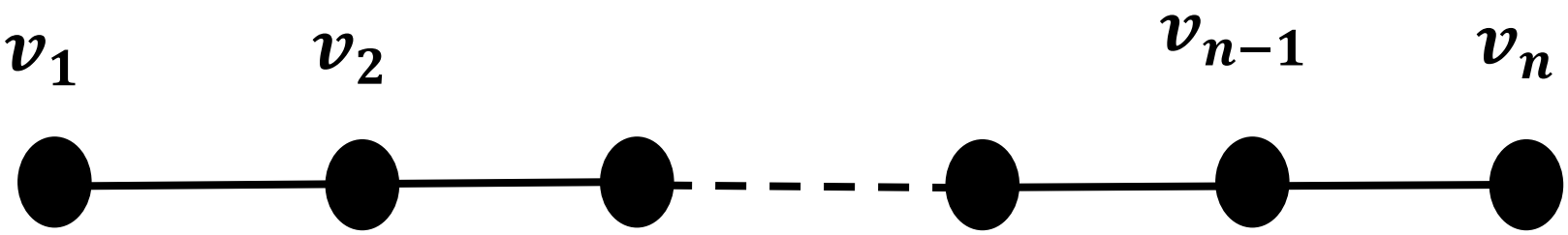
Finite Graph:
both sets of vertices and edges are finite

Graph G

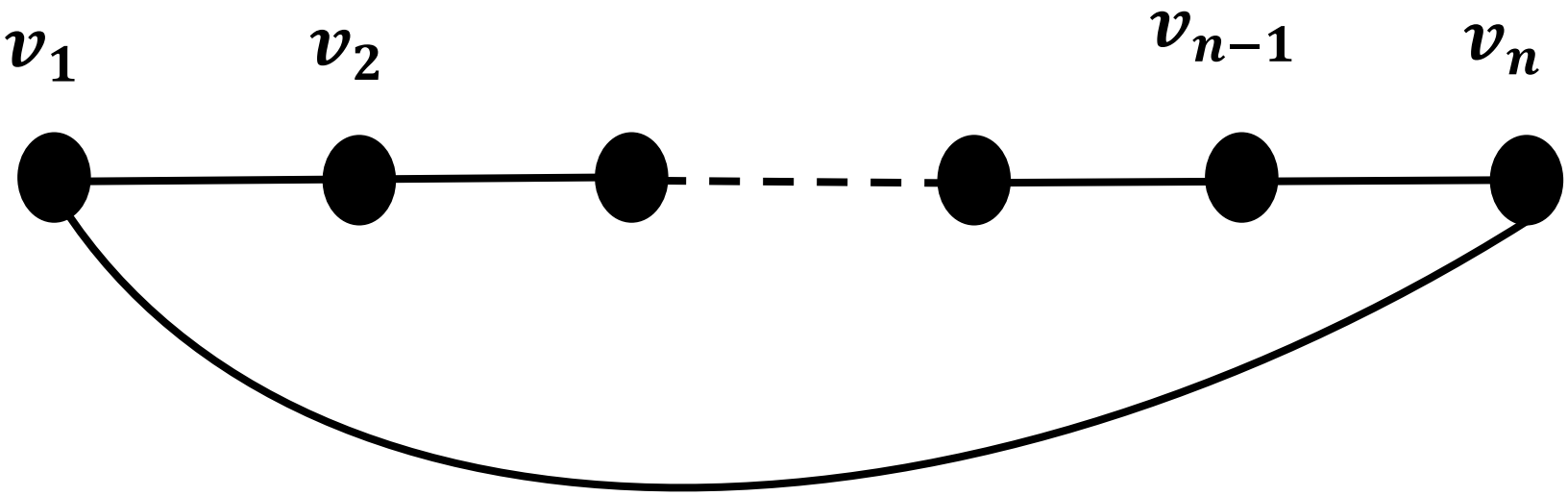


Definitions

path



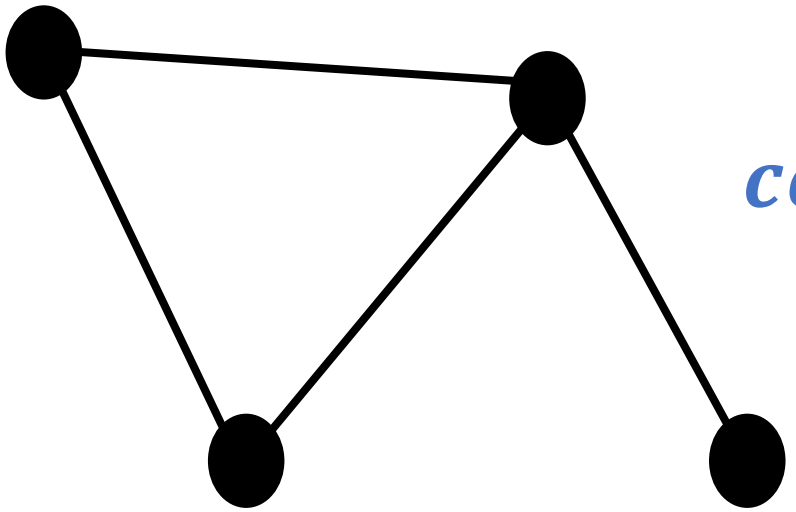
cycle



Definitions

***connected graph*: any 2 vertices are joined by a path**

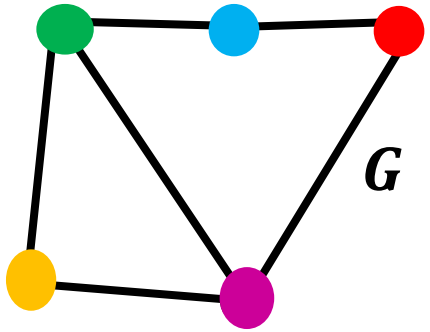
Example



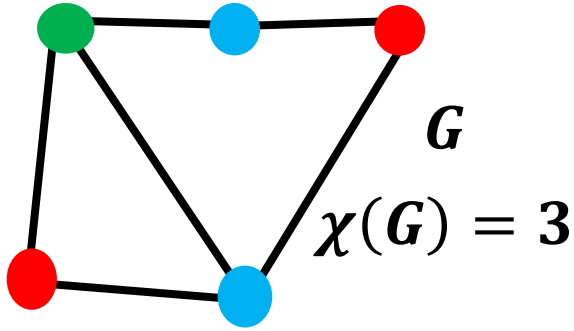
connected

Chromatic number of a Graph

A proper coloring of a graph G is an assignment of colors to the vertices of G, such that any two adjacent vertices have different colors.



The chromatic number of a graph G, denoted by $\chi(G)$, is the smallest integer k such that G has a proper coloring with k colors.



Definitions

$V(D)$: set of vertices

$E(D)$: set of arcs

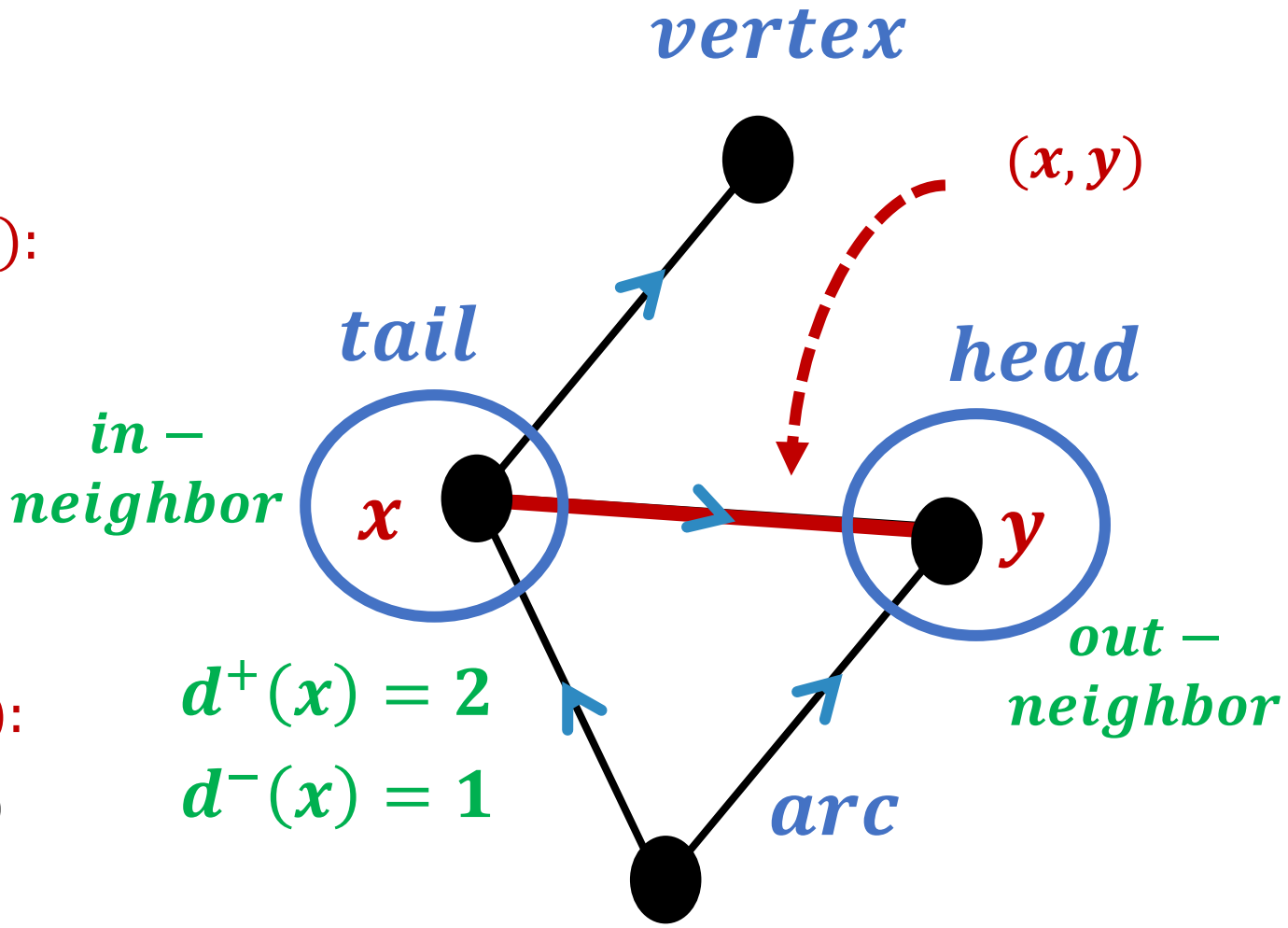
Digraph D

Out − neighbor (resp. in − neighbor) of a vertex v denoted by $N^+(v)$ (resp. $N^-(v)$):

set of heads (resp. tails) adjacent to v

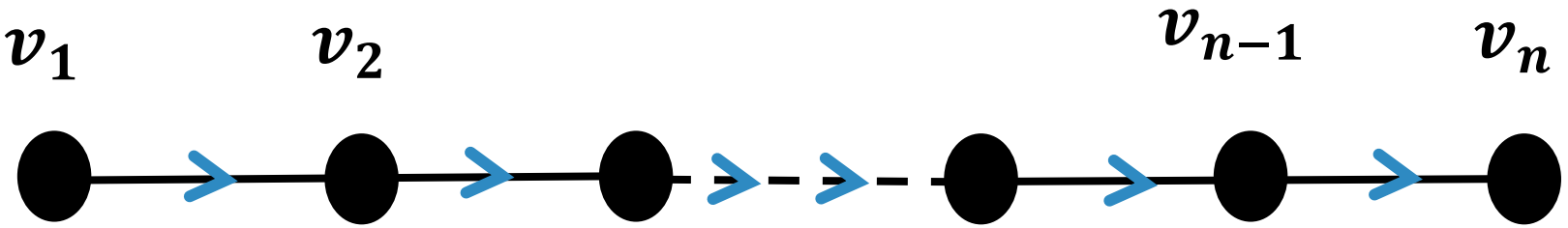
Out − degree (resp. in − degree) of a vertex v denoted by $d^+(v)$ (resp. $d^-(v)$):

number of heads (resp. tails) adjacent to v

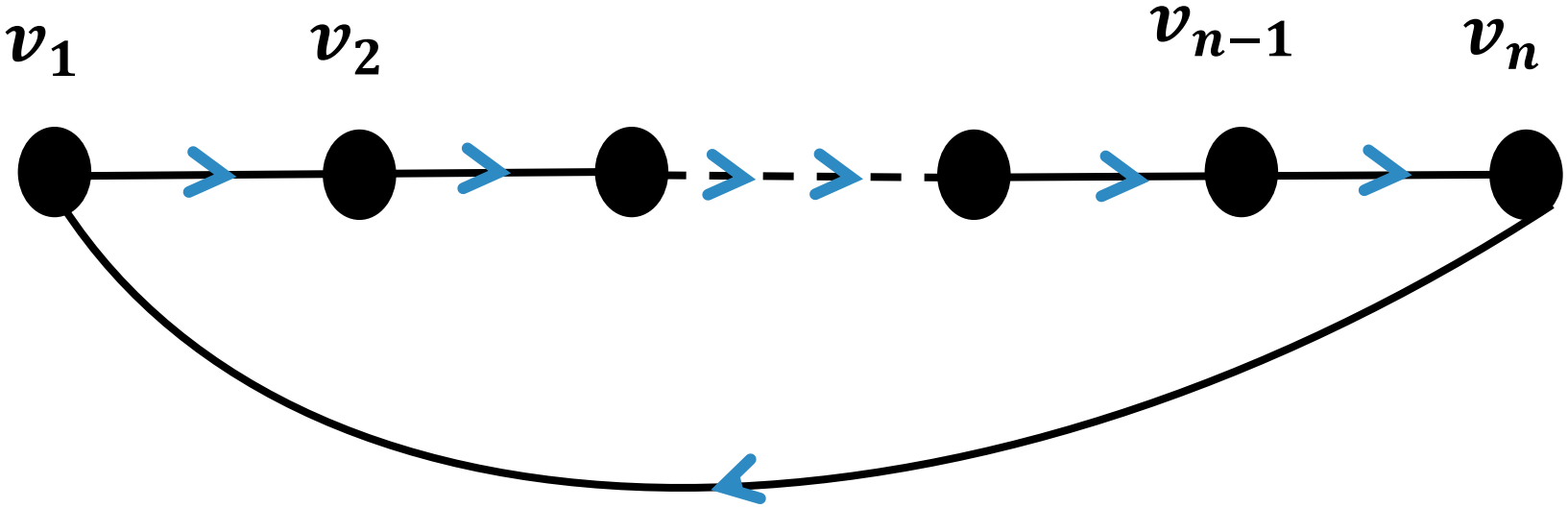


Definitions

directed path



circuit

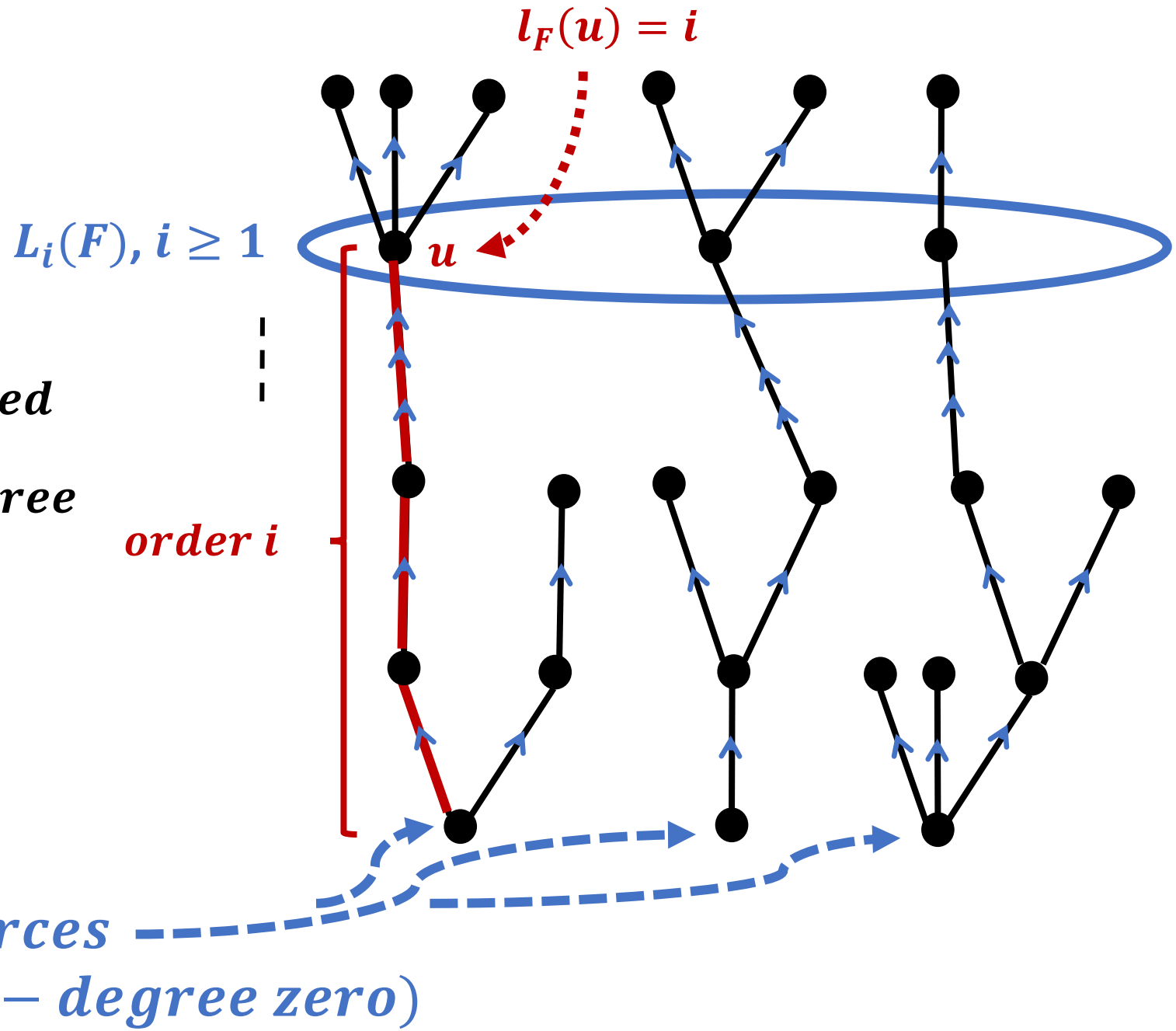


Remark

For any digraph D, $|V(D)|$ ***is the order of D***

Outforest

1. *each component is connected*
2. *all vertices are of in − degree one at most*
3. *free of circuits*



Outforest

Theorem

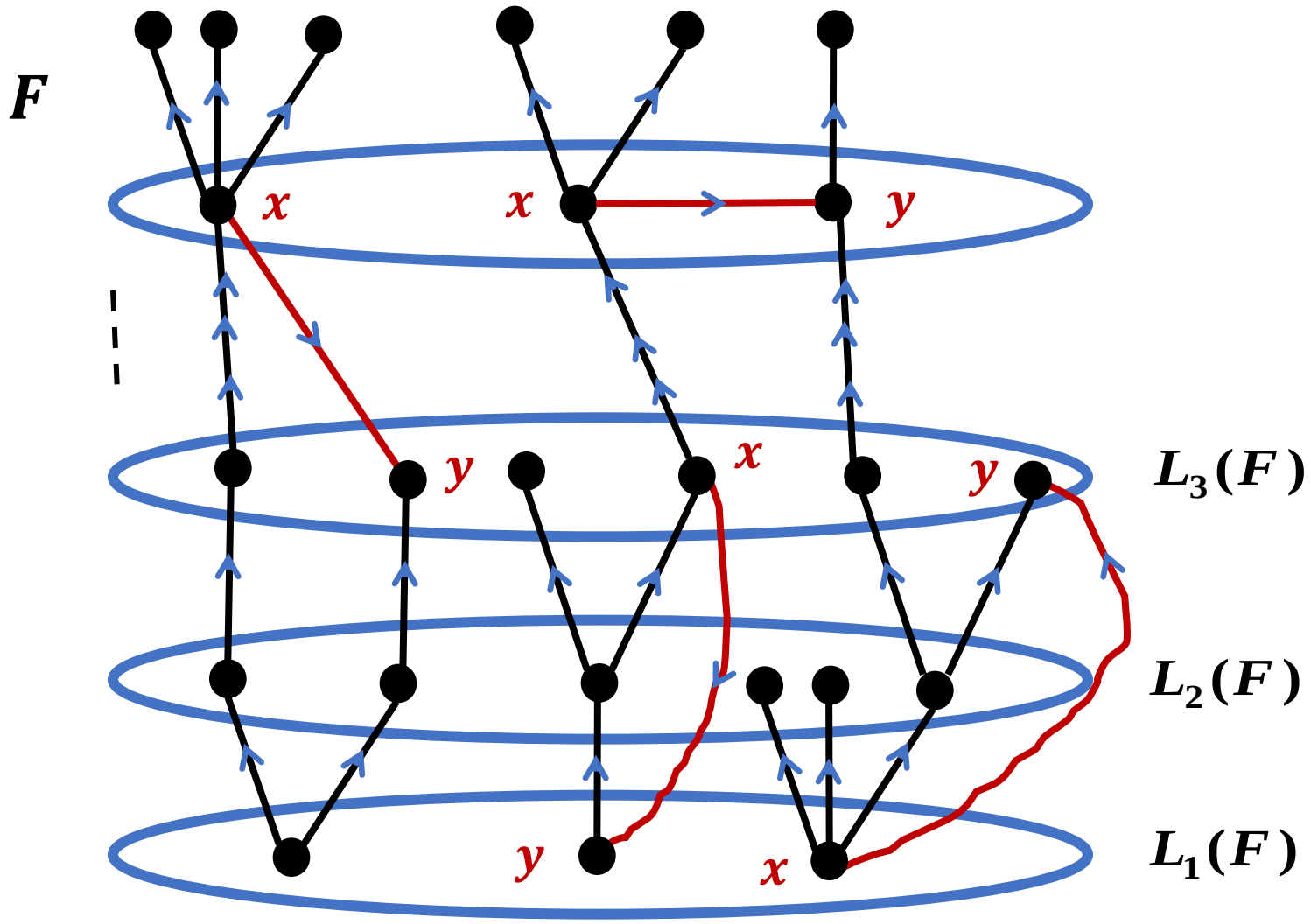
Every digraph contains a spanning outforest.

Outforest

D: digraph, F: spanning outforest of D

$(x, y) \in E(D)$

- forward arc
- backward arc
  - good arc
  - bad arc



Maximal Outforest

$Definition(El\ Sahili - Kouider)$

Consider a digraph D, and let F be a spanning outforest of D.
F is said to be a maximal outforest of D if:

- $V(F) = V(D)$
- ***There are no bad arcs with respect to F***

Maximal Outforest

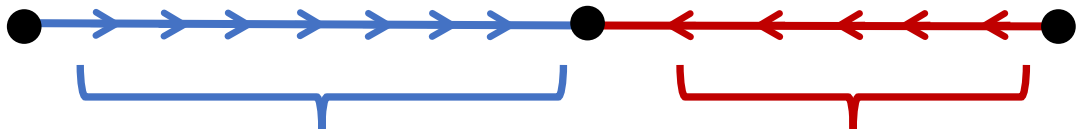
$Theorem(El\ Sahili - Kouider)$

Consider a digraph D, and let F be a maximal outforest of D. Then,

- $L_i(F)$ ***is stable*** $\forall\ i \geq 1$
- $\chi(D) \leq l(F)$ ***where*** $l(F)$ ***is the maximum integer*** i ***such that*** $L_i(F) \neq \phi$

Path with two Blocks

$P(k,l)$



k forward arcs l backward arcs

Conjecture(El Sahili)

Every $n-$ chromatic digraph, $n \geq 4$, contains a $P(k,l)$ with $k+l=n-1$.

$k=0 \longrightarrow Roy-Gallai$

$k=1 \longrightarrow Bondy-El\ Sahili$

Path with two Blocks

$\boldsymbol{Case\ of\ study: k \geq 2}$

$\boldsymbol{1st\ proof\ of\ El\ Sahili\ Conjecture:}$

$\boldsymbol{L.Addario-Berry, F.Havet\ and\ Thomassé}$

Path with two Blocks

$\boldsymbol{Case\ of\ study}$: $\boldsymbol{k \geq 2}$

$\boldsymbol{New\ Elementary\ Proof}$

$\boldsymbol{k - minimal\ outforest}$

$\boldsymbol{Characterization\ of\ a\ graph\ containing\ no\ P(k, l)}$

k −minimal Outforest, $k \geq 2$

$\boldsymbol{D}$: $\boldsymbol{digraph}$

$\boldsymbol{F}$: $\boldsymbol{spanning\ outforest}$

$$u_k(F) = \sum_{i=1}^{k} |L_i(F)|$$

Definition

F is said to be a $k-$ minimal outforest if and only if $u_k(F)$ is minimal.

New Elementary Proof of the Path with two Blocks Conjecture

$D: n-chromatic\ digraph$
$D \not\supseteq P(k,l); k+l=n-1$

$F: k-minimal\ outforest, k \geq 2$

$U_i(F) = L_i(F)\ \forall\ 1 \leq i \leq k-1$

$$U_i(F) = \bigcup_{\alpha \geq 0} L_{i+\alpha(l+1)}(F)\ \forall\ k \leq i \leq k+l$$

Conclusion: $\chi(D) \leq n-1$ ✘

Thank You