Stabilization of the one dimensional wave equation with localized internal Kelvin-Voigt type damping

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Introduction

We consider the following wave equation:

$$\begin{aligned} u_{tt} - au_{xx} &= 0 \qquad 0 < x < 1, t > 0, && (0.1) \\ u(0,t) = u(1,t) &= 0 \qquad t > 0 && (0.2) \end{aligned}$$

with the following initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \; 0 < x < 1 \qquad (0.3)$$

where a is a positive constant.

Introduction

The natural energy of system (1.1)-(1.3) is given by

$$E(t) = \frac{1}{2}\int_0^1 \left(|u_t|^2 + |u_x|^2\right) dx. \tag{0.4}$$

Then a straightforward computation gives

$$\frac{d}{dt}E(t) = 0. \tag{0.5}$$

Then system (1.1)-(1.3) is conservative in the sense that the energy $E(t)$ is constant.

Introduction

We consider the following wave equation:

$$\begin{aligned} u_{tt} - a u_{xx} + c(x) u_t &= 0 \qquad 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) &= 0 \qquad t > 0 \end{aligned} \tag{1.1}$$

with the following initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),\ 0 < x < 1 \tag{1.2}$$

where a is a positive constant and $c \in C^0([0,1], \mathbb{R}^+)$.

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where a is a positive constant and $c \in C^0([0,1],\mathbb{R}^+)$.

$$E(t) = \frac{1}{2}\int_0^1 (|u_t|^2 + |u_x|^2)dx. \tag{1.3}$$

$$\frac{d}{dt}E(t) = -\int_0^1 c(x)|u_t|^2 dx. \tag{1.4}$$

Then the system is dissipative in the sense that its energy is decreasing.

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$$E(t) \leq \frac{c}{t^{\alpha}} ||U_0||^2_{D(\mathcal{A})}, \quad \forall t > 0.$$
- **Logarithmic Stability:** $\exists\ c\ , \alpha > 0$ **such that**
$$E(t) \leq \frac{c}{(ln(1+t))^{\alpha}} ||U_0||^2_{D(\mathcal{A})}, \quad \forall t > 0.$$

Introduction

Consider the following elastic wave equation with local Kelvin -Voigt damping

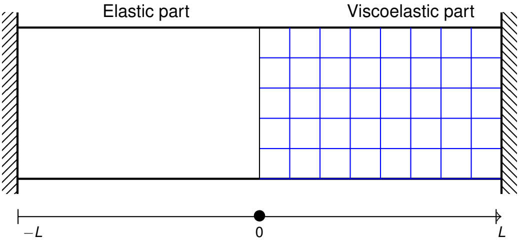
$$\begin{cases} \rho\, U_{tt}(x,t) - [\kappa\, U_x(x,t) + D(x)\, U_{xt}(x,t)]_x = 0, & (x,t) \in (0,L) \times (0\ +\infty), \\ U(0,t) = U(L,t) = 0, & t \in (0\ +\infty), \\ (U(x,0), U_t(x,0)) = (U_0(x), U_1(x))\,, & x \in (0,L). \end{cases} \tag{1.5}$$

Introduction

In 1998, K. Liu and Z. Liu in [3] considered a wave equation with localized Kelvin-Voigt damping in the 1-dimensional case. The dissipation is distributed on any subinterval of the region occupied by the beam and the damping coefficient is the characteristic function of the subinterval. They proved that the semigroup associated with the equation for the longitudinal motion of the beam is not exponentially stable. This result is due to the discontinuity of the viscoelastic materials.

Introduction

In 2013, Alves et al. in [2] considered a model with only two parts where one part is elastic and the other is viscoelastic with Kelvin-Voigt constitutive relation. Under the assumption that the two materials are distributed equally, they established an optimal polynomial decay rate of the solution of type t^{-2}.



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In 2014, Alves et al. in [1] considered the transmission problem of a material composed of three components; a Kelvin-Voigt viscoelastic material and two elastic materials where one of them is affected by frictional damping and the second is without any dissipation.

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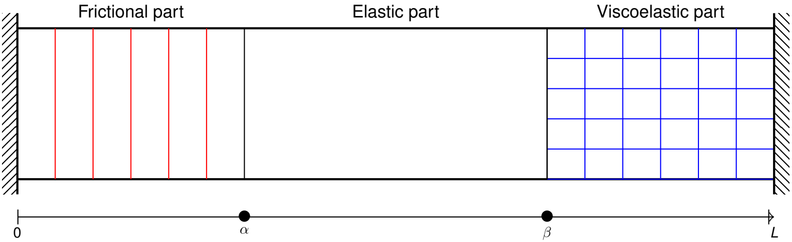

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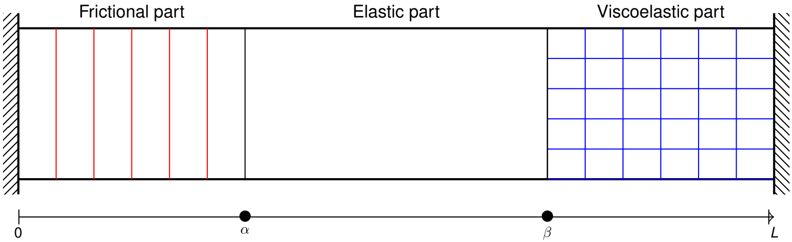

$\Longrightarrow$ EXPONENTIAL STABILITY

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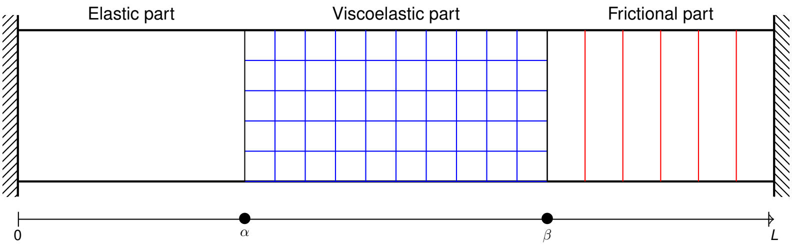

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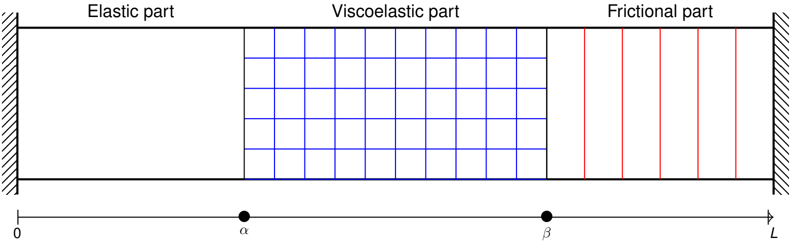

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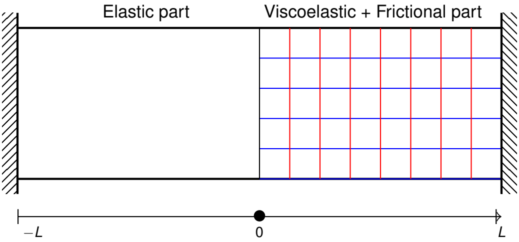
Introduction

In 2017, Oquendo in [4] considered two transmission problems with two damping mechanisms.



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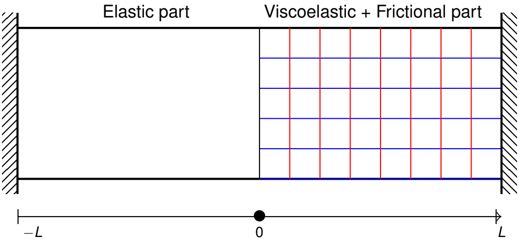

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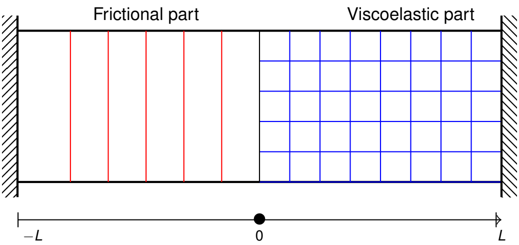

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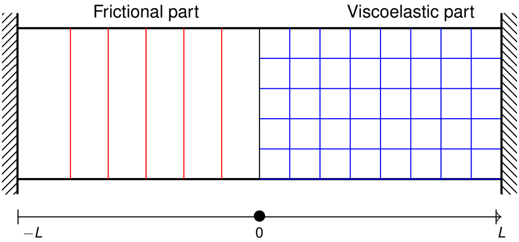

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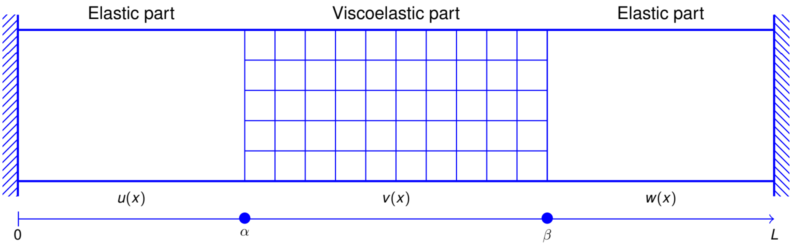

Introduction

$$U = \begin{cases} u(x), & x \in (0, \alpha), \\ v(x), & x \in (\alpha, \beta), \\ w(x), & x \in (\beta, L). \end{cases}$$

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In this case, System (1.5) is equivalent to the following system

$$\rho_1 u_{tt}-\kappa_1 u_{xx}=0,\quad (x,t)\in(0,\alpha)\times(0,+\infty), \tag{1.6}$$

$$\rho_2 v_{tt}-\kappa_2 v_{xx}-\delta v_{xxt}=0,\quad (x,t)\in(\alpha,\beta)\times(0,+\infty), \tag{1.7}$$

$$\rho_3 w_{tt}-\kappa_3 w_{xx}=0,\quad (x,t)\in(\beta,L)\times(0,+\infty), \tag{1.8}$$

with the Dirichlet boundary conditions

$$u(0,t)=w(L,t)=0,\quad t\in(0,+\infty). \tag{1.9}$$

Introduction

The transmission conditions are given by

$$\begin{cases} u(\alpha,t) = v(\alpha,t), \quad v(\beta,t) = w(\beta,t), & t \in (0,+\infty), \\ \kappa_2 v_x(\alpha,t) + \delta v_{xt}(\alpha,t) = \kappa_1 u_x(\alpha,t), & t \in (0,+\infty), \\ \kappa_2 v_x(\beta,t) + \delta v_{xt}(\beta,t) = \kappa_3 w_x(\beta,t), & t \in (0,+\infty). \end{cases} \tag{1.10}$$

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System (1.6)-(1.10) is subjected to the following initial conditions

$$\begin{cases} (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in (0,\alpha), \\ (v(x,0), v_t(x,0)) = (v_0(x), v_1(x)), & x \in (\alpha,\beta), \\ (w(x,0), w_t(x,0)) = (w_0(x), w_1(x)), & x \in (\beta,L). \end{cases} \tag{1.11}$$

Introduction

The energy of solutions of the System (1.6)-(1.11) is defined by:

$$E(t) = \frac{\kappa_1}{2}\int_0^{\alpha} |u_x|^2 dx + \frac{\rho_1}{2}\int_0^{\alpha} |u_t|^2 dx$$
$$+\frac{\kappa_2}{2}\int_{\alpha}^{\beta} |v_x|^2 dx + \frac{\rho_2}{2}\int_{\alpha}^{\beta} |v_t|^2 dx$$
$$+\frac{\kappa_3}{2}\int_{\beta}^{L} |w_x|^2 dx + \frac{\rho_3}{2}\int_{\beta}^{L} |w_t|^2 dx.$$

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$$+\frac{\kappa_2}{2}\int_\alpha^\beta |v_x|^2dx+\frac{\rho_2}{2}\int_\alpha^\beta |v_t|^2dx$$
$$+\frac{\kappa_3}{2}\int_\beta^L |w_x|^2dx+\frac{\rho_3}{2}\int_\beta^L |w_t|^2dx.$$

A direct computation gives

$$\frac{d}{dt}E(t)=-\delta\int_\alpha^\beta |v_{xt}|^2\,dx\leq 0.$$

Thus System (1.6)-(1.11) is dissipative in the sense that its energy is non increasing with respect to the time t.

Well posedness

For well-posedness, let us define

$$\mathbb{H}^m = H^m(0,\alpha) \times H^m(\alpha,\beta) \times H^m(\beta,L), \quad m = 1,2,$$

$$\mathbb{L}^2 = L^2(0,\alpha) \times L^2(\alpha,\beta) \times L^2(\beta,L),$$

$$\mathbb{H}^1_L = \{(u,v,w) \in \mathbb{H}^1 \mid u(0) = w(L) = 0,\ u(\alpha) = v(\alpha),\ v(\beta) = w(\beta)\}.$$

The Hilbert energy space is given by

$$\mathcal{H} = \mathbb{H}^1_L \times \mathbb{L}^2$$

Well posedness

We define the linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ by:

$$D(\mathcal{A}) = \{(u, v, w, y, z, \phi) \in \mathcal{H} \mid (y, z, \phi) \in \mathbb{H}_L^1,\ (u, \kappa_2 v + \delta z, w) \in \mathbb{H}^2,$$
$$\kappa_2 v_x(\alpha) + \delta z_x(\alpha) = \kappa_1 u_x(\alpha),\ \kappa_2 v_x(\beta) + \delta z_x(\beta) = \kappa_3 w_x(\beta)\}$$

and for all $\mathbb{U} = (u, v, w, y, z, \phi) \in D(A)$

$$\mathcal{A}\mathbb{U} = \left(y, z, \phi, \frac{\kappa_1}{\rho_1} u_{xx}, \frac{1}{\rho_2}(\kappa_2 v_{xx} + \delta z_{xx}), \frac{\kappa_3}{\rho_3} w_{xx} \right).$$

Well posedness

If (u, v, w, u_t, v_t, w_t) is a regular solution of System (1.6)-(1.11), then we transform this system into the following initial value problem

$$\begin{cases} \mathbb{U}_t & = \mathcal{A}\mathbb{U}, \\ \mathbb{U}(0) & = \mathbb{U}_0, \end{cases} \tag{1.12}$$

where $\mathbb{U}_0 = (u_0, v_0, w_0, u_1, v_1, w_1) \in \mathcal{H}$.

Existence and Uniqueness

Thanks to Lumer-Philips theorem, we deduce that $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in $\mathcal{H}$ and therefore Problem (1.6)-(1.11) is well-posed. Then we have the following result:

Existence and Uniqueness

Thanks to Lumer-Philips theorem, we deduce that $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in $\mathcal{H}$ and therefore Problem (1.6)-(1.11) is well-posed. Then we have the following result:

Theorem: Existence and uniqueness "Ghader - Nasser - Wehbe"

For any $\mathbb{U}_0 \in \mathcal{H}$, Problem (1.12) admits a unique weak solution

$$\mathbb{U}(t) = e^{t\mathcal{A}}\mathbb{U}_0 \in C^0(\mathbb{R}^+, \mathcal{H}).$$

Moreover, if $\mathbb{U}_0 \in D(\mathcal{A})$, then

$$\mathbb{U}(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).$$

Strong stability

Theorem: Strong stability "Ghader - Nasser - Wehbe"

The C_0-semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable on the Hilbert space $\mathcal{H}$ in the sense that

$$\lim_{t\to+\infty} ||e^{t\mathcal{A}}\mathbb{U}_0||_{\mathcal{H}} = 0, \qquad \forall\ \mathbb{U}_0 \in \mathcal{H}.$$

Non-uniform stability

Theorem: Non-uniform stability "Ghader - Nasser - Wehbe"

Let $\frac{p}{q}$ $(p,\ q \in \mathbb{Z}^*)$ be an irreducible fraction. Assume that

$$\frac{L-\beta}{\alpha} = \frac{p}{q}\sqrt{\frac{\rho_1 \kappa_3}{\rho_3 \kappa_1}}.$$

For $\epsilon > 0$ (small enough), we cannot expect the energy decay rate $t^{-4-\epsilon}$ for all initial data $\mathbb{U}_0 \in D(\mathcal{A})$ and for all $t > 0$.

Polynomial stability and optimality

Theorem: Polynomial Stability "Ghader - Nasser - Wehbe"

For all initial data $\mathbb{U}_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of $\mathbb{U}_0$ such that the energy of System (1.6)-(1.11) satisfies the following estimation

$$E(t) \leq \frac{C}{t^4} ||\mathbb{U}_0||^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{4.1}$$

In addition, if

$$\frac{L - \beta}{\alpha} = \frac{p}{q} \sqrt{\frac{\rho_1 \kappa_3}{\rho_3 \kappa_1}}, \tag{4.2}$$

then the energy decay rate in (4.1) is optimal.

Sketch of the proof

Using the result of Borichev and Tomilov, (4.1) holds if and only if:

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{H1}$$

$$\sup_{\beta\in\mathbb{R}} \|(i\beta I - \mathcal{A})^{-1}\|_{L(\mathcal{H})} = O\left(|\beta|^{\frac{1}{2}}\right). \tag{H2}$$

Sketch of the proof

Suppose that $(H2)$ is false. Then, there exists $\beta_n \in \mathbb{R}$ and a sequence $\mathbb{U}_n \in D(A)$ such that

$$|\beta_n| \to +\infty, \quad ||\mathbb{U}_n||_{\mathcal{H}} = ||(u_n, v_n, w_n, y_n, z_n, \phi_n)||_{\mathcal{H}} = 1, \tag{4.3}$$

$$\lambda_n^{\frac{1}{2}}(i\lambda_n I - \mathcal{A})\mathbb{U}_n = (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}, f_{6,n}) \to 0 \text{ in } \mathcal{H}. \tag{4.4}$$

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$$\lambda_n^{\frac{1}{2}}(i\lambda_n I - \mathcal{A})\mathbb{U}_n = (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}, f_{6,n}) \to 0 \text{ in } \mathcal{H}. \tag{4.4}$$

We need to show that $||\mathbb{U}_n||_{\mathcal{H}} \to 0$ and hence we get a contradiction to (4.3).

Sketch of the proof

Let $g \in C^1\left([\alpha, \beta]\right)$ such that

$$g(\beta) = -g(\alpha) = 1, \quad \max_{x\in[\alpha,\beta]} |g(x)| = c_g \quad \text{and} \quad \max_{x\in[\alpha,\beta]} |g'(x)| = c_{g'},$$

where c_g and $c_{g'}$ are strictly positive constant numbers.

Introduction

- We prove the following asymptotic behavior estimate

$$|z(\beta)|^2 + |z(\alpha)|^2 \leq \left(\frac{\rho_2 \lambda^{\frac{1}{2}}}{2} + 2c_{g'} \right) \int_\alpha^\beta |z|^2 \, dx + o\left(\lambda^{-1}\right). \quad (4.5)$$

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- We prove the following asymptotic behavior estimate

$$|\kappa_2 v_x(\alpha) + \delta z_x(\alpha)|^2 + |\kappa_2 v_x(\beta) + \delta z_x(\beta)|^2 \leq \frac{\rho_2 \lambda^{\frac{3}{2}}}{2} \int_\alpha^\beta |z|^2 dx + o(1). \quad (4.6)$$

Sketch of the proof

- We prove the following asymptotic behavior estimations

$$\int_{\alpha}^{\beta} |z|^2 dx = o\left(\lambda^{-\frac{3}{2}}\right), \tag{4.7}$$

$$|z(\alpha)|^2 = o\left(\lambda^{-1}\right), \quad |z(\beta)|^2 = o\left(\lambda^{-1}\right), \tag{4.8}$$

$$|\kappa_2 v_x(\alpha) + \delta z_x(\alpha)| = o(1), \quad |\kappa_2 v_x(\beta) + \delta z_x(\beta)| = o(1). \tag{4.9}$$

Sketch of the proof

We prove the following asymptotic behavior estimations

$$\int_0^{\alpha} |y|^2 dx = o(1), \quad \int_0^{\alpha} |u_x|^2 dx = o(1) \tag{4.10}$$

and

$$\int_{\beta}^{L} |\phi|^2 dx = o(1), \quad \int_{\beta}^{L} |w_x|^2 dx = o(1). \tag{4.11}$$

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From the above results, we get

$$\|U\|_{\mathcal{H}} = o(1),$$

which contradicts the fact that

$$\|U\|_{\mathcal{H}} = 1.$$

M. Alves, J. M. Rivera, M. Sepúlveda, and O. V. Villagrán.
The Lack of Exponential Stability in Certain Transmission Problems with Localized Kelvin–Voigt Dissipation.
SIAM Journal on Applied Mathematics, 74(2):345–365, Jan. 2014.

M. Alves, J. M. Rivera, M. Sepúlveda, O. V. Villagrán, and M. Z. Garay.
The asymptotic behavior of the linear transmission problem in viscoelasticity.
Mathematische Nachrichten, 287(5-6):483–497, Oct. 2013.

K. Liu and Z. Liu.
Exponential Decay of Energy of the Euler–Bernoulli Beam with Locally Distributed Kelvin–Voigt Damping.
SIAM Journal on Control and Optimization, 36(3):1086–1098, may 1998.

H. P. Oquendo.
Frictional versus Kelvin-Voigt damping in a transmission problem.
Mathematical Methods in the Applied Sciences,
40(18):7026–7032, July 2017.

Thank you for Attention!