# Well-posedness and exponential stability of a thermoelastic system with internal delay.

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# Introduction

## Motivation

Let us consider the following thermoelastic system with delay

$$\begin{cases} u_{tt}(x,t) - \alpha u_{xx}(x,t-\tau) + \gamma\theta_x(x,t) = 0 & \text{in } (0,\ell)\times(0,\infty), \\ \theta_t(x,t) - \kappa\theta_{xx}(x,t) + \gamma u_{xt}(x,t) = 0 & \text{in } (0,\ell)\times(0,\infty), \\ u(0,t) = u(\ell,t) = \theta_x(0,t) = \theta_x(\ell,t) = 0, & t \geq 0 \end{cases} \quad (1)$$

$\alpha, \gamma$ and $\ell$ : Positive constants. $u = u(x,t)$ :The displacement ; $\theta = \theta(x,t)$ :The temperature difference,$x \in (0,\ell)$,$t \geq 0$ ,$\tau > 0$ :The time delay.

## Motivation

Racke proved that, under some initial and boundary conditions, the system (1) is not well posed and unstable even if $\tau$ is relatively small. However, it is well known that, in the absence of delay, the damping through the heat conduction is strong enough to produce an exponential stable system .

R.Racke
Instability of of coupled systems with delay.
*Commun.Pure Appl.Anal.* **11** (2012), 1753-1773.

## Motivation

In recent years, the PDEs with time delays effects became an active area of research. In fact, time delays so often arise in many applications since, most physical phenomena not only depend on the present state but also on some past occurrences, but as for the classical thermoelastic system, an arbitrary small delay may destroy the well-posedness of the problem or may destroy the stability.
In order to solve such problems, additional conditions or control terms have been used,

## Somme examples

### Example 1 :Interior delay

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a_0 u_t(x,t) + a u_t(x,t-\tau) = 0, \quad \text{in } \Omega \times (0,+\infty,) \\ u(x,t) = 0, \; x \in \Gamma_0, t > 0, \\ \frac{\partial u}{\partial \nu}(x,t) = 0, \; x \in \Gamma_1, t > 0, \end{cases} \tag{2}$$

S. Nicaise and C. Pignotti
Stability and instability results of the wave equation with a delay tem in the boundary or internal feedbacks,
*SIAM J .Control Optim.*, **45(5)** ,(2006), 1561-1585.

## Example 2 :Boundary delay.

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & x \in \Gamma_0, t > 0, \\ \frac{\partial u}{\partial \nu}(x,t) = -a_0 u_t(x,t) - a u_t(x,t-\tau) & x \in \Gamma_1, t > 0, \end{cases} \tag{3}$$

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Nicaise and Pignotti examined this two systems and proved under the assumption $a < a_0$,that the energy is exponentially stable.

S. Nicaise and C. Pignotti
Stability and instability results of the wave equation with a delay tem in the boundary or internal feedbacks,
*SIAM J .Control Optim.*, **45(5)** ,(2006), 1561-1585.

## Example 3 :Distributed delay

$$\begin{cases} au_{tt}(x,t) - du_{xx}(x,t) + \beta\theta_x(x,t) = 0, \text{ in } \Omega \times (0,\infty) \\ b\theta_t(x,t) - k_1\theta_{xx}(x,t) - \int_{\tau_1}^{\tau_2} k_2(s)\theta_{xx}(x,t-s)ds + \beta u_{xt}(x,t) = 0, in\ \Omega \times (0,\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x), \quad x \in \Omega \\ \theta_x(x,-t) = f_0(x,t), \quad x \in \Omega, \quad t \in (0,\tau_2) \end{cases}$$

Under the condition $\int_{\tau_1}^{\tau_2} |k_2(s)|\, ds < k_1$,the damping effect through heat conduction is strong enough to uniformly stabilize the system.

Muhammad I.Mustafa and Mohammad.Kafini
Exponential Decay in Thermoelastic System with Internal Distributed delay.
*Palestine Journal of Mathematics.* **2(2)** (2013), 287-299.

## Stratigy

## Kelvin-Voigt damping

We add a Kelvin-Voigt damping of the form $-\beta u_{xxt}(x,t)$ ,$\beta > 0$ .

$$\begin{cases} u_{tt}(x,t) - \alpha u_{xx}(x,t-\tau) - \beta u_{xxt}(x,t) + \gamma\theta_x(x,t) = 0, & \text{in } \Omega \times (0,\infty), \\ \theta_t(x,t) - \kappa\theta_{xx}(x,t) + \gamma u_{xt}(x,t) = 0, & \text{in } \Omega \times (0,\infty), \\ u(0,t) = u(\ell,t) = 0, & \text{in } (0,\infty), \\ \theta_x(0,t) = \theta_x(\ell,t) = 0, & \text{in } (0,\infty), \\ u_x(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times (0,\tau), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x), & \text{in } \Omega \end{cases} \tag{4}$$

$\Omega = (0,\ell)$ and $(u_0, u_1, \theta_0, f_0)$ belong to a suitable space.

## Stratigy

### Kelvin-Voigt damping

Ammari et al added a Kelvin-Voigt damping term to the abstract equation.

$$\begin{cases} \ddot{u}(t) + aBB^*\dot{u}(t) + BB^*u(t-\tau) = 0, & \text{in } (0,\infty), \\ u(0) = u_0, \quad \dot{u}(0) = u_1, & \\ B^*u(t-\tau) = f_0(t-\tau), & \text{in } (0,\tau), \end{cases} \tag{5}$$

where $B : \mathcal{D}(B) \subset H_1 \to H$ is a linear unbounded operator, $H_1$ ,$H$, are two Hilbert spaces,$a > 0$ and $B^*$ satisfies some properties of coercivity and compactness embedding.
$\rightsquigarrow$ Exponential decay result under the assumption $\tau \leq a$.

K. Ammari, S. Nicaise and C. Pignotti
Stability of abstract-wave equation with delay and Kelvin-Voigt damping
*Asymptotic. Anal.* **95** (2015), 21-38.

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1. Introduction
2. Well-posedness
3. Exponential stability

# Well-posedness

Let us define the energy of a solution of problem (4) as

$$E(t) := \frac{1}{2}\int_{\Omega}\left(u_t^2(x,t) + u_x^2(x,t) + \theta^2(x,t)\right)dx + \frac{\xi}{2}\int_{\Omega}\int_0^1 u_x^2(x,t-\tau\rho)d\rho dx$$

where $\xi > 0$ is a parameter fixed later on.

## Equivalente system

⇝New variable :

$$z(x,\rho,t)=u_x(t-\tau\rho),\ (x,\rho,t)\in\Omega\times(0,1)\times(0,+\infty). \tag{6}$$

⇝Equivalent system of (4) :

$$\begin{cases} u_{tt}(x,t)-\alpha z_x(x,1,t)-\beta u_{xxt}(x,t)+\gamma\theta_x(x,t)=0, & \text{in } \Omega\times(0,\infty),\\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0, & \text{in } \Omega\times(0,1)\times(0,+\infty),\\ \theta_t(x,t)-\kappa\theta_{xx}(x,t)+\gamma u_{xt}(x,t)=0, & \text{in } \Omega\times(0,\infty),\\ u(0,t)=u(\ell,t)=0, & \text{in } (0,\infty),\\ \theta_x(0,t)=\theta_x(\ell,t)=0, & \text{in } (0,\infty),\\ z(x,0,t)=u_x(x,t), & \text{in } \Omega\times(0,\infty),\\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), \theta(x,0)=\theta_0(x), & \text{in } \Omega,\\ z(x,\rho,0)=f_0(x,-\tau\rho), & \text{in } \Omega\times(0,\tau). \end{cases} \tag{7}$$

## Hilbet space

$$\mathcal{H} = \left\{ (f, g, p, h) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1)) \times L^2(\Omega) \mid \int_\Omega h(x)dx = 0 \right\}.$$

$\rightsquigarrow \forall U_k = (f_k, g_k, h_k, p_k)^T \in \mathcal{H}, \quad k = 1, 2;$
$\langle U_1, U_2 \rangle_{\mathcal{H}} = \int_\Omega (\alpha f_{1x}(x) f_{2x}(x) + g_1(x) g_2(x) + h_1(x) h_2(x))\, dx +$
$\frac{\xi}{\tau} \int_\Omega \int_0^1 p_1(x,\rho) p_2(x,\rho)\, d\rho dx.$
$\rightsquigarrow \mathcal{H}$ is an Hilbert space.

## Cauchy Problem

Let $v = u_t$ ; define $U := (u, v, z, \theta)^T$
$\rightsquigarrow$ Problem (7) can be formulated as :

$$\begin{cases} U' = \mathcal{A}U \\ U(0) = (u_0, u_1, f_0(., -.\tau), \theta_0)^T \end{cases} \tag{8}$$

## Cauchy Problem

$\rightsquigarrow$ $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ z \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ (\alpha z(.,1) + \beta v_x)_x - \gamma\theta_x \\ -\frac{1}{\tau} z_\rho \\ -\gamma v_x + \kappa\theta_{xx} \end{pmatrix}.$$

## Domaine of $\mathcal{A}$

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{c} U = (u, v, z, \theta) \in \mathcal{H} \cap \left[H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega; H^1(0,1)) \times H^2(\Omega)\right] \mid \\ z(.,0) = u_x \quad \text{and} \quad (\alpha z(.,1) + \beta v_x) \in H^1(\Omega) \end{array} \right.$$

**Theorem**

For any initial datum $U_0 \in \mathcal{H}$ there exists a unique solution $U \in \mathcal{C}([0,+\infty),\mathcal{H})$ of problem (8). Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then $U \in \mathcal{C}([0,+\infty),\mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1([0,+\infty),\mathcal{H})$.

## Proof

Take $U = (u, v, z, \theta)^T \in D(A)$. We choose $\xi = \frac{2\tau\alpha^2}{\beta}$ to get for $a^* = \frac{2\alpha^2}{\beta}$,

$$\langle (\mathcal{A} - a^* Id)\, U, U \rangle_{\mathcal{H}} \leq -\frac{\beta}{2} \int_{\Omega} v_x^2(x) dx - \kappa \int_{\Omega} \theta_x^2 \, dx.$$

$\longrightarrow \mathcal{A} - a^* Id$ is dissipative.

Maximality of $\mathcal{A} - a^* Id$

$\rightsquigarrow \lambda \mathrm{I} - \mathcal{A}$, is surjective for a fixed $\lambda > a^*$.

Given $(f, g, p, h)^T \in \mathcal{H}$, we look for a solution $U = (u, v, z, \theta)^T \in D(\mathcal{A})$ of

$$(\lambda \mathrm{I} - \mathcal{A}) \begin{pmatrix} u \\ v \\ z \\ \theta \end{pmatrix} = \begin{pmatrix} f \\ g \\ p \\ h \end{pmatrix}$$

Wich gives : For all $w \in H_0^1(\Omega)$ and $\varphi \in H^2(\Omega)$ such that $\varphi_x(0) = \varphi_x(l) = 0$

$$\lambda^2 \int_\Omega uwdx + \left(\alpha e^{-\lambda\tau} + \lambda\beta\right) \int_\Omega u_x w_x dx + \gamma \int_\Omega \theta_x w dx = \int_\Omega (g + \lambda f) w dx + \int_\Omega (f_x - \alpha z_0) w_x dx$$

and

$$\lambda \int_\Omega \theta\varphi dx - \lambda\gamma \int_\Omega u\varphi_x dx + \kappa \int_\Omega \theta_x \varphi_x dx = \int_\Omega (h - \gamma f)\varphi dx.$$

$\rightsquigarrow$ Formulation

$$b\left((u,v),(w,\varphi)\right) = F(w,\varphi) \tag{9}$$

where

$b\left((u,v),(w,\varphi)\right) = \int_\Omega \left[\lambda^2 uw + \left(\alpha e^{-\lambda\tau} + \lambda\beta\right) u_x w_x\right] dx +$
$\int_\Omega \left(\theta\varphi + \frac{\kappa}{\lambda}\theta_x\varphi_x\right) dx + \gamma \int_\Omega \left(\theta_x w - u\varphi_x\right) dx$

and

$$F(w,\varphi) = \int_\Omega (g+\lambda f) w dx + \int_\Omega (f_x - \alpha z_0) w_x dx + \int_\Omega (h - \gamma f)\varphi dx.$$

The space $\mathcal{F} := \left\{(w,\varphi) \in H_0^1\Omega) \times H^2(\Omega) \mid \varphi_x(0) = \varphi_x(l) = 0\right\}$,
equipped with :
$\langle (w_1,\varphi_1),(w_2,\varphi_2)\rangle_{\mathcal{F}} = \int_\Omega (w_1 w_2 + w_{1x} w_{2x} + \varphi_1\varphi_2 + \varphi_{1x}\varphi_{2x})\, dx,$
is an Hilbert space.
$\rightsquigarrow$ $b$ on $\mathcal{F}\times\mathcal{F}$ and $F$ on $\mathcal{F}$ are continuous.
$\rightsquigarrow$ for every $(w,\varphi)\in\mathcal{F}$, $|b((w,\varphi),(w,\varphi))| \geq c\|(w,\varphi)\|^2_{\mathcal{H}}$
with $c := \min\left(\lambda^2, \left(\alpha e^{-\lambda\tau} + \lambda\beta\right), 1, \frac{\kappa}{\lambda}\right) > 0.$
Lax-Milgram lemma, $\rightsquigarrow$ (9) has a unique solution $(u,\theta)\in\mathcal{F}$.
$\rightsquigarrow$ $A - a^* Id$ generates a $\mathcal{C}_0$-semigroup of contraction.
The bounded perturbation theorem $\rightsquigarrow$ $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup on $\mathcal{H}$
$\rightsquigarrow$ The well-posedness follows from semigroup theory.

# Plan

# Exponential stability

**Theorem**

There exists $\beta_0 > 0$ such that for every $\beta \geq \beta_0$, the associated system is exponentially stable.

## Proof

### Lyapunov fonction

$$V(t) := \frac{N_1}{2} V_1(t) + \alpha \frac{N_2}{2} V_2(t) + \frac{N_3}{2} V_3(t) + N_4 V_4(t) + N_5 V_5(t) + N_6 V_6(t).$$

where

$$\begin{aligned} V_1(t) &:= \|u_t\|^2, \quad V_2(t) := \|u_x\|^2, \quad V_3(t) := \|\theta\|^2, \\ V_4(t) &:= \int_0^1 \varphi^2(\rho) \|z(., \rho)\|^2 d\rho, \; V_5(t) = -\int_0^1 \varphi(\rho) f(\rho) \langle z(., \rho), u_x \rangle d\rho, \\ V_6(t) &:= \langle u, u_t \rangle, \end{aligned}$$

and $N_1, N_2, N_3, N_4, N_5$ and $N_6$ are positive numbers to be fixed later. $\varphi$ is a decreasing positive function continuously derivable on $[0, 1]$ s.t $|\varphi'| \geq k_1 \varphi$ for $k_1 > 0$, $f$ is a real function defined on $[0, 1]$ : determined later to.

Let $V\tilde{(}t)$ the energy defined by :

$$\tilde{V}(t) := \frac{N_1}{2} V_1(t) + \alpha \frac{N_2}{2} V_2(t) + \frac{N_3}{2} V_3(t) + N_4 V_4(t).$$

$\rightsquigarrow$ $\tilde{V}(t) \simeq E(t)$.
$\rightsquigarrow$ for a suitable choice of $\varphi$, $g$ and $f$ we can find $\{N_1, ..., N_6\}$ and $\beta > 0$ s.t :

(A1) $V(t)$ is equivalent to $\tilde{V}(t)$.

(A2) $\tilde{V}'(t) \leq -n_0 \tilde{V}(t)$, for $n_0 > 0$.

By taking $N_3 = N_1$, $N_6\beta = N_2\alpha$ and $N_6\alpha = \frac{f(1)\varphi(1)}{\tau}N_5$
For all $\varepsilon_1, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7 > 0$ .

$$\begin{aligned}
\dot{V(t)} \leq & \left(-N_4\frac{\varphi^2(1)}{\tau} + N_1\frac{\alpha\varepsilon_1}{2}\right)\|z(.,1)\|^2 \\
+ & \left(-2\frac{k_1}{\tau}N_4 + \frac{N_5}{2\tau}\left(\varepsilon_3 + \frac{\tau}{\varepsilon_4}\right)\right)V_4(t) \\
+ & \left(N_4\frac{\varphi^2(0)}{\tau} + \frac{N_5}{\tau}\left(\frac{\Gamma}{2\varepsilon_3} - \Lambda\right) + N_6\frac{\gamma\varepsilon_5 c_p}{2}\right)\|u_x\|^2 \\
+ & \left(N_1\left(\frac{\alpha}{2\varepsilon_1} - \beta\right) + N_5\frac{\varepsilon_4}{2}\Phi + N_6 c_p\right)\|u_{tx}\|^2 \\
+ & \left(-N_3\kappa + N_6\frac{\gamma}{2\varepsilon_5}\right)\|\theta_x\|^2
\end{aligned}$$

Where $c_p > 0$.

$$c := f(1)\varphi(1), \;\; \Lambda := f(0)\varphi(0) = c + \int_0^1 g(\rho)\varphi(\rho)d\rho, \;\; \Phi := \int_0^1 f^2(\rho)d\rho,$$

and

$$\Gamma := \int_0^1 g^2(\rho)d\rho$$

with

$$-g\varphi = \varphi' f + \varphi f'. \quad g > 0..$$

- $N_4 = aN_1$ with $a = \frac{1}{2}\alpha\varepsilon_1\frac{\tau}{\varphi^2(1)}, .$
- $N_5 = bN_4 = abN_1$ with $b = \frac{4kk_1}{\varepsilon_3 + \frac{\tau}{\varepsilon_4}}, \quad 0 < k < 1.$
- $N_6 = \frac{c}{\alpha T}N_5 = \frac{abc}{\alpha T}N_1$
- $N_2 = \frac{\beta}{\alpha}N_6 = \frac{abc\beta}{\alpha^2 T}N_1$

Recall that $V(t)$, $\tilde{V}(t)$ and $E(t)$ are equivalent then, there exists $a_0 > 0,\ C > 0$ suth that

$$E(t) \leq Ce^{-a_0 t}$$

Thank you