A priori estimates of solutions for fractional differential equations with boundary integral conditions

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Preliminaries

We need some definitions and lemmas to explain the problem that we shall study in this work., the Caputo derivative and fractional integral are, respectively, defined as follows

Definition (1)

For $0 < \alpha < 1$,the left Caputo derivatives can be expressed as

$$ {}_0^C\partial_t^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial u}{\partial \tau}(x,\tau)}{(t-\tau)^{\alpha}} d\tau \tag{1} $$

Definition (2)

For $0 < \alpha < 1$,the fractional integral is defined by

$$ I_t^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(x,\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{2} $$

Corollary (1)

For $0 < \alpha < 1$*, the composition of Caputo derivative and integral operator is given by*

$$\int_0^t {}_0^C\partial_\tau^\alpha u d\tau = I_t^{1-\alpha} u - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} u(x,0) \quad (3)$$

Lemma (1)

(**Poincare type inequality**) *For all* $u \in L^2(Q)$ *and* $m, n \in \mathbb{N}$ *sach that* $m \leq n$ *we have :*

$$\int_0^1 (\Im_x^n u)^2 dx \leq \frac{1}{2^{n-m}} \int_0^1 (\Im_x^m u)^2 dx. \quad (4)$$

Lemma (1)

where

$$\begin{cases} \Im_x^n u = \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1} u(\xi,t)\, d\xi, \quad n \geq 1 \\ \Im_x^0 u = u(x,t) \end{cases} \tag{5}$$

Lemma (2)

For $0 < \alpha < 1$ *and for all continu function* u *on* Q *we have*

$$\left({}_0^C\partial_t^\alpha u\right) u \geq \frac{1}{2} \left({}_0^C\partial_t^\alpha u\right)^2 . \tag{6}$$

Formulation of the problem

In the rectangular domain $Q = (0,1) \times (0,T)$, with $T < \infty$. and $0 < \alpha < 1$, we consider the FDE :

$$\mathcal{L}v = \left({}_0^c\partial_t^\alpha v\right) + (-1)^m \frac{\partial^{2m-1}}{\partial x^{2m-1}}\left(a(x,t)\frac{\partial v}{\partial x}\right) = g\,(x,t)\,, m \in \mathbb{N}^*, \quad (7)$$

subject to the initial condition

$$\ell v = v(x,0) = \varphi(x), x \in (0,1)\,, \quad (8)$$

and the purely integral boundary conditions

$$\int_0^1 \xi^i v\,(\xi,t)\,d\xi = e_i(t),\ \ i = \overline{0, 2m-1},\ \ t \in (0,T) \quad (9)$$

where $a(x,t)$, $g(x,t)$, $\varphi(x)$, , $e_i(t)$ $\left(i=\overline{0,2m-1}\right)$ are given functions that satisfy certain conditions which will be specified later and $\frac{\partial^{2m-1}a(x,t)}{\partial x^{2m-1}} \in L^2(Q)$. The fucntion $\varphi(x)$ satisfied some compatibilty conditions given as follows

$$\int_0^1 \xi^i \varphi(\xi)\, d\xi = e_i(0),\ \ i=\overline{0,2m-1}. \tag{10}$$

First, we transform problem $(7)-(9)$ with inhomogenous integral condition, to the equivalent problem with homogenous integral boundary conditions.

We introduce a new unknown function $u(x,t)$ defined by

$$u(x,t) = v(x,t) - U(x,t), \tag{11}$$

where

$$U(x,t) = \sum_{j=0}^{2m-1} b_j(t) x^j. \tag{12}$$

and

$$\begin{aligned} A &= \left(\frac{1}{1+i+j}\right)_{2m\times 2m}, B=\left(b_j\left(t\right)\right)_{2m\times 1}, \text{ and} \\ E &= \left(e_i(t)\right)_{2m\times 1}, \ 0\leq i,j\leq 2m-1, \end{aligned}$$

then

$$B = A^{-1}E.$$

Problem $(7)-(9)$ equivalent to

$$\mathcal{L}u = \left({}_0^c\partial_t^\alpha u\right) + (-1)^m \frac{\partial^{2m-1}}{\partial x^{2m-1}}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f\left(x,t\right), \tag{13}$$

$$\ell v = u(x,0) = \psi(x), \tag{14}$$

$$\int_0^1 \xi^i v\left(\xi,t\right) d\xi = 0, \ \ i = \overline{0,2m-1} \tag{15}$$

where

$$\begin{aligned} f(x,t) &= g(x,t) - \mathcal{L}U(x,t), \\ \psi(x) &= \varphi(x) - \ell U. \end{aligned}$$

From a compatibilty conditions (10), we obatin

$$\int_0^1 \xi^i \psi(\xi)\, d\xi = e_i(0), \ \ i = \overline{0, 2m-1}.$$

Hence, the solution of problem $(7) - (9)$ will be obtained by the equations. $(11) - (12)$

Existence and uniqueness of the solutions

A priori estimate method (So called energy-integral method) is one efficient functional analysis technique. This is an important technique for studying PDEs in general and with some purely integral conditions.
First, the problem $(13)-(15)$ is equivalent to the operational equation

$$Lu = F,$$

where $L = (\mathcal{L}, \ell)$ is considered as an operator from B to H, B is a Banach space consistiong of all function $u \in L^2(Q)$ with the finite norm :

$$\|u\|_B^2 = \sup_{0\leq\tau\leq T} \left(I_\tau^{1-\alpha}\left(\int_0^1 (\Im_x^m u)^2\, dx\right) + \Im_\tau\left(\int_0^1 (\Im_x u)^2\, dx\right)\right)$$

and H is the Hilbert space constiting of all elements $F = (f, \psi)$ with the finite norm :

$$\|F\|_H^2 = \int_Q f^2 dxdt + \int_0^1 \psi^2 dx$$

The domain of definition $D(L)$ is the set of all functions $u \in L^2(Q)$ for which ${}_0^C\partial_t^\alpha u$, $\frac{\partial^{2m} u}{\partial x^{2m}} \in L^2(Q)$ and satisfying integral conditions (15)

Definition (3)

The operator L from B into H has a closure $\overline{L}$. The solution of the operational equation

$$\overline{L}u = F$$

is called strong solution of the problem $(13) - (15)$.

Theorem (1)

For all functions $a(x,t)$ satisfying condition :

$$2\inf_Q a - \frac{1}{2}\sup_Q \frac{\partial a}{\partial x} \geq a_0 > 0, \tag{16}$$

the solution of problem $(13) - (15)$ satisfies a priori estimate

$$\|u\|_B \leq K \|F\|_H , \quad \forall u \in D(L)$$

Démonstration.

We take the scalar product in $L^2(0,1)$ of equation (13) and the integrodifferential operator $Mu = (-1)^m \Im_x^{2m} u$ we have

$$\langle \mathcal{L}u, Mu \rangle_{L^2(0,1)} = \langle f, Mu \rangle_{L^2(0,1)}$$

then

$$\begin{aligned} & (-1)^m \int_0^1 \left({}_0^C\partial_t^\alpha u \right) \Im_x^{2m} u dx + \int_0^1 \frac{\partial^{2m-1}}{\partial x^{2m-1}} \left(a(x,t) \frac{\partial u}{\partial x} \right) \Im_x^{2m} u dx \\ = & \ (-1)^m \int_0^1 f \ \Im_x^{2m} u dx. \end{aligned} \tag{17}$$

Integrating by parts the left hand side of equation (17) and using integral conditions (15) give

$$(-1)^m \int_0^1 \left({}_0^C\partial_t^\alpha u \right) \Im_x^{2m} u dx = \int_0^1 \left({}_0^C\partial_t^\alpha \Im_x^m u \right) \Im_x^m u dx \tag{18}$$

□

Démonstration.

and

$$\int_0^1 \frac{\partial^{2m-1}}{\partial x^{2m-1}} \left(a(x,t) \frac{\partial u}{\partial x} \right) \Im_x^{2m} u dx = \int_0^1 \left(-\frac{1}{2} \frac{\partial a}{\partial x} \left(\Im_x u \right)^2 + a u^2 \right) dx \tag{19}$$

Substuting equations (18) and (19) in equation (17), we obtain

$$\left\{ \begin{array}{c} \int_0^1 \left({}_0^C\partial_t^\alpha \Im_x^m u \right) \Im_x^m u dx + \int_0^1 \left(-\frac{1}{2} \frac{\partial a}{\partial x} \left(\Im_x u \right)^2 \right) dx + \int_0^1 a u^2 dx \\ = (-1)^m \int_0^1 f \Im_x^{2m} u dx. \end{array} \right. \tag{20}$$

We use , **Lemma 1**, the second and third term of left hant side of equation (20) estimate by

$$\left\{ \begin{array}{c} \int_0^1 \left({}_0^C\partial_t^\alpha \Im_x^m u \right) \Im_x^m u dx + \left(2 \inf_Q a(x,t) - \frac{1}{2} \sup_Q \frac{\partial a}{\partial x} \right) \int_0^1 \left(\Im_x u \right)^2 dx \\ \leq (-1)^m \int_0^1 f \Im_x^{2m} u dx. \end{array} \right.$$

□

Démonstration.

by **Lemma 2** and condition (16), we deduce

$$\frac{1}{2}\int_0^1 {}_0^C\partial_t^\alpha \left(\Im_x^m u\right)^2 dx + a_0 \int_0^1 u^2 dx \leq (-1)^m \int_0^1 f.\Im_x^{2m} u dx, \tag{21}$$

The right hand side of equation (21), Cauchy $\varepsilon-$inequality and **Lemma 1** give :

$$\left|(-1)^m \int_0^1 f.\Im_x^{2m} u dx\right| \leq \frac{1}{2\varepsilon}\int_0^1 f^2 dx + \frac{\varepsilon}{2^{2m}}\int_0^1 \left(\Im_x u\right)^2 dx. \tag{22}$$

Substuting inequality (22) in estimate (21) we obtain

$$\frac{1}{2}\int_0^1 {}_0^C\partial_t^\alpha \left(\Im_x^m u\right)^2 dx + \left(a_0 - \frac{\varepsilon}{2^{2m}}\right)\int_0^1 \left(\Im_x u\right)^2 dx \leq \frac{1}{2\varepsilon}\int_0^1 f^2 dx. \tag{23}$$

□

Démonstration.

Integrating inequality (23) over $(0, \tau)$ and using the **Corollary 1,** then

$$\frac{1}{2} I_{\tau}^{1-\alpha}\left(\int_0^1 \left(\Im_x^m u\right)^2 dx\right) + \left(a_0 - \frac{\varepsilon}{2^{2m}}\right) \Im_{\tau}\left(\int_0^1 \left(\Im_x u\right)^2 dx\right)$$
$$\leq \frac{1}{2\varepsilon}\int_0^{\tau}\int_0^1 f^2 dxdt + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\int_0^1 \psi^2 dx \qquad (24)$$

therfore

$$\frac{1}{2} I_{\tau}^{1-\alpha}\left(\int_0^1 \left(\Im_x^m u\right)^2 dx\right) + \left(a_0 - \frac{\varepsilon}{2^{2m}}\right) \Im_{\tau}\left(\int_0^1 \left(\Im_x u\right)^2 dx\right)$$
$$\leq \frac{1}{2\varepsilon}\int_0^{T}\int_0^1 f^2 dxdt + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\int_0^1 \psi^2 dx \qquad (25)$$

□

Démonstration.

We choose ε such that $a_0 - \frac{\varepsilon}{2^{2m}} \geq 0$ then

$$\frac{1}{2} I_\tau^{1-\alpha} \left(\int_0^1 \left(\Im_x^m u\right)^2 dx \right) + \Im_\tau \left(\int_0^1 \left(\Im_x u\right)^2 dx \right)$$

$$\leq \quad K^2 \left(\int_Q f^{\,2} dxdt + \int_0^1 \psi^2 dx \right) \tag{26}$$

where

$$K^2 = \left(\frac{\max\left(\frac{1}{2\varepsilon}, \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\right)}{\min\left(\frac{1}{2}, a_0 - \frac{\varepsilon}{2^{2m}}\right)} \right)$$

□

Démonstration.

Finally

$$\sup \left(I_{\tau}^{1-\alpha} \left(\int_{Q} \left(\Im_{x}^{m} u\right)^{2} dx \right) \right) + \Im_{\tau} \left(\int_{0}^{1} \left(\Im_{x} u\right)^{2} dx \right)$$
$$\leq \quad K^{2} \left(\int_{Q} f^{2} dxdt + \int_{0}^{1} \psi^{2} dx \right)$$

then

$$\|u\|_{B} \leq K \|Lu\|_{H} , \tag{27}$$

Since $\overline{L}$ is the closure of L, we can extend inequality(27)as follows

$$\|u\|_{B} \leq K \left\|\overline{L}u\right\|_{H} , \quad \forall u \in D(\overline{L}) \tag{28}$$

Hence, inequality (28) leads to the following corollaries : □

Corollary (2)

A strong solution of $(13)-(15)$ *is unique if it existe, and depends continuously on* $F=(f,\psi)$.

Corollary (3)

The rang of L is closed in H and $R(L)=\overline{R(L)}$.

We proved the uniqueness of solution, if is a solution. However, we have not demonstrated the existence yet. To do it, we will just prove that $R(L)$ is dense in H.

Theorem (2)

Let us suppose that the conditions of Theorem 1 *are filled, for* $\omega\in L^2(Q)$ *and* $u\in D_0(L)=\{u\in D(L), lu=0\}$,*we have*

$$\int_Q \mathcal{L}u\omega\, dxdt=0, \tag{29}$$

then ω *vanishes almost everywhere in* Q.

Démonstration.

We can rewrite equation (29) as follows

$$\int_Q \left({}_0^C\partial_t^\alpha u\right)\omega dxdt + (-1)^m \int_Q \frac{\partial^{2m-1}}{\partial x^{2m-1}}\left(a(x,t)\frac{\partial u}{\partial x}\right)\omega dxdt = 0 \quad (30)$$

We express ω in terms of a function u as

$$\omega = \Im_x^{2m} u. \quad (31)$$

Substuting ω in (30) give

$$\int_Q \left({}_0^C\partial_t^\alpha u\right)\Im_x^{2m} udxdt + (-1)^m \int_Q \frac{\partial^{2m-1}}{\partial x^{2m-1}}\left(a(x,t)\frac{\partial u}{\partial x}\right)\Im_x^{2m} udxdt = 0 \quad (32)$$

Integrating by parts all terms of left hand side over $(0,1)$ in (32) and using integral conditions (15) we get

$$\int_Q \left({}_0^C\partial_t^\alpha u\right)\Im_x^{2m} udxdt = (-1)^m \int_Q \left({}_0^C\partial_t^\alpha \Im_x^m u\right)\Im_x^m udxdt, \quad (33)$$

Démonstration.

and

$$\left\{ \begin{array}{c} (-1)^m \int_Q \frac{\partial^{2m-1}}{\partial x^{2m-1}} \left(a(x,t) \frac{\partial u}{\partial x} \right) \Im_x^{2m} u dx dt = \\ (-1)^m \int_Q \left(-\frac{1}{2} \frac{\partial a}{\partial x} \left(\Im_x u \right)^2 + a u^2 \right) dx dt \end{array} \right. \tag{34}$$

Substuting (33) and (34) in (32), we obtain

$$\int_Q \left({}_0^C\partial_t^\alpha \Im_x^m u \right) \Im_x^m u dx dt + \int_Q a u^2 dx dt - \frac{1}{2} \int_Q \frac{\partial a}{\partial x} \left(\Im_x u \right)^2 dx dt = 0. \tag{35}$$

In other hand by using **Lemma 1** we give

$$\int_0^1 \left({}_0^C\partial_t^\alpha \Im_x^m u \right) \Im_x^m u dx + \int_0^1 a u^2 dx - \frac{1}{2} \int_0^1 \frac{\partial a}{\partial x} \left(\Im_x u \right)^2 dx \geq$$

$$\int_0^1 \left({}_0^C\partial_t^\alpha \Im_x^m u \right) \Im_x^m u dx + \left(2 \inf_Q a - \frac{1}{2} \sup_Q \frac{\partial a}{\partial x} \right) \int_0^1 \left(\Im_x u \right)^2 dx \tag{36}$$

After we use conditions (16) and **Lemma 2**, we have □

Démonstration.

$$\int_0^1 \left({}_0^C\partial_t^\alpha \Im_x^m u\right) \Im_x^m u dx + \int_0^1 a u^2 dx - \frac{1}{2}\int_0^1 \frac{\partial a}{\partial x} \left(\Im_x u\right)^2 dx \geq$$

$$\frac{1}{2}\int_0^1 \left({}_0^C\partial_t^\alpha \Im_x^m u\right)^2 dx + a_0 \int_0^1 \left(\Im_x u\right)^2 dx \qquad (37)$$

Integrating inequality (37) over $(0,\tau)$ and using the **Corollary 1,** give

$$\int_0^\tau \int_0^1 \left({}_0^C\partial_t^\alpha \Im_x^m u\right) \Im_x^m u dxdt + \int_0^\tau \int_0^1 a u^2 dxdt - \frac{1}{2}\int_0^\tau \int_0^1 \frac{\partial a}{\partial x} \left(\Im_x u\right)^2 dxdt \geq$$

$$\frac{1}{2} I_\tau^{1-\alpha} \left(\int_0^1 \left(\Im_x^m u\right)^2 dx\right) + a_0 \int_0^\tau \int_0^1 \left(\Im_x u\right)^2 dxdt$$

We replace τ by T,we conclude that □

Démonstration.

$$\int_Q \left({}_0^C\partial_t^\alpha \Im_x^m u\right) \Im_x^m u dxdt + \int_Q au^2 dxdt - \frac{1}{2}\int_Q \frac{\partial a}{\partial x} \left(\Im_x u\right)^2 dxdt \geq$$

$$\frac{1}{2} I_T^{1-\alpha} \left(\int_0^1 \left(\Im_x^m u\right)^2 dx\right) + a_0 \int_Q \left(\Im_x u\right)^2 dxdt,$$

since

$$\int_Q \left({}_0^C\partial_t^\alpha \Im_x^m u\right) \Im_x^m u dxdt + \int_Q au^2 dxdt - \frac{1}{2}\int_Q \frac{\partial a}{\partial x} \left(\Im_x u\right)^2 dx = 0,$$

hence

$$\left(\Im_x u\right)^2 = 0$$

therfore $u = 0$, then $\omega \equiv 0$ in $L^2(Q)$. □

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Thank you for your attention!