Exponential stability for the nonlinear Schrödinger equation on a star-shaped network

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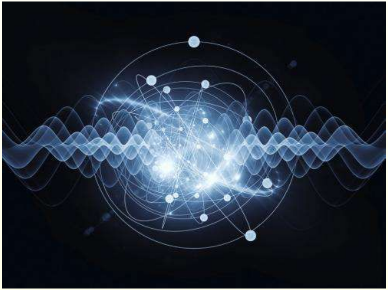
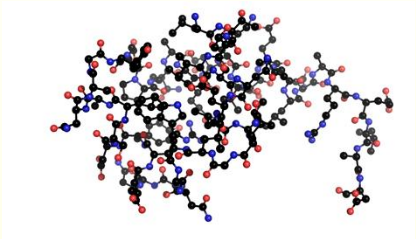
Outline

Some physical motivations





The NSE on a star-shaped network, $\mathcal{R}$

We are interested in Schrödinger equation in an infinite graph $\mathcal{R}$:

$$(E)\begin{cases} i\,\partial_t u_1 + \partial_x^2 u_1 + \lambda u_1|u_1|^{\alpha-1} + ia(x)u_1 = 0 \text{ for } \; x>0,\ t>0, \\ i\,\partial_t u_i + \partial_x^2 u_i + \lambda u_i|u_i|^{\alpha-1} = 0 \text{ for } x>0,\ t>0,\ 2\le i\le N, \\ \sum_{i=1}^{N} \partial_x u_i(t,0) = 0,\ t>0, \\ u_i(t,0) = u_j(t,0),\ \forall\, t>0,\ 1\le i,j\le N, \\ u_i(0,x) = \varphi_i(x),\ \ x>0,\ \ 1\le i\le N, \end{cases}$$

where a satisfies the following condition :

$$(H)\begin{cases} a\in L^\infty(\mathbb{R}_+) \text{is non-negative function almost everywhere and,} \\ a(x)\ge\alpha_0>0 \quad \text{for } |x|>R,\ R>0. \end{cases}$$

The NSE on a star-shaped network, $\mathcal{R}$

- $u : \mathbb{R} \times \mathcal{R} \longrightarrow \mathbb{C}$ is a complex-valued function of time and space.
- Duhamel formula :

$$\begin{cases} u_1(t) & = e^{it\partial_x^2}\varphi_1 + i\int_0^T e^{i\partial_x^2(t-s)}\left(\lambda|u_1|^{\alpha-1}u_1 + i\,a(.)u_1\right)(s)ds, \\ u_i(t) & = e^{it\partial_x^2}\varphi_i + i\int_0^T e^{i\partial_x^2(t-s)}\left(\lambda|u_i|^{\alpha-1}u_i\right)(s)ds\ ,\ 2 \leq i \leq N. \end{cases}$$

- The energy identity :

$$E_u(t) := -2\int_0^t \int_{\mathbb{R}_+} a(x)|u_1(s,x)|^2\,dxds + E_u(0),\ \forall\, t \geq 0,$$

where $u = (u_1, ..., u_N)$ and $E_{u_i}(t)$ is the energy of u_i, $1 \leq i \leq N$, defined by $E_{u_i}(t) := \displaystyle\int_0^{+\infty} |u_i(x,t)|^2\,dx$.

Our main goal : Show the exponential decay of the global energy at the infinity.

Previous works

Exponential decay rate of the energy in L^2-level [Natali], L^∞-time decay rate in a star shaped network [Banica-Ignat, Ali Mehmeti-Ammari-Nicaise].

F. Natali, A note on the exponential decay for the nonlinear Schrödinger equation, *Osaka J. Math.*, **53** (2016), 717–729.

F. Natali, Exponential stabilization for the nonlinear Schrödinger equation with localized damping, *J. Dyn. Control Syst.*, **21** (2015), 461–474.

V. Banica, L.I. Ignat, *Dispersion for the Schrödinger equation on networks, J. Math. Phys.* **52** (2011) 083703.

F. Ali Mehmeti, K. Ammari and S. Nicaise, Dispersive effects and high frequency behaviour for the Schrödinger equation in star-shaped networks, *Port. Math.*, **72** (2015), 309–355.

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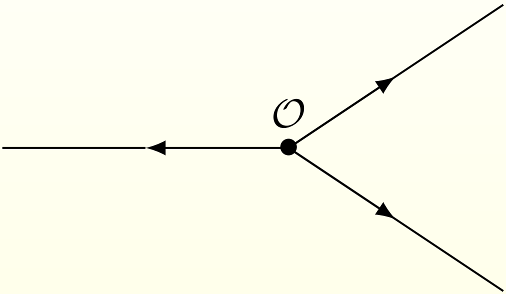
Preliminaries on a star-shaped network



Preliminaries on a star-shaped network

The considered graph $\mathcal{R}$ consists of a finite number of edges, of infinite length attached to a common vertex, each of them being identified with a copy of the positive semi-axis.
Let $\mathcal{R}_i$, $i = 1, 2, ..., N$ be N disjoint sets identified with to $(0, +\infty)$. We set $\mathcal{R} := \overset{N}{\underset{i=1}{\cup}} \overline{\mathcal{R}_k}$. We denote by $f = (f_k)_{k=1,2,...,N} = (f_1, f_2, ..., f_N)$ the functions on $\mathcal{R}$ taking their values in $\mathbb{C}$ and let f_k be the restriction of f to $\mathcal{R}_k$. Define the Hilbert space $\mathcal{H} = \overset{N}{\underset{i=1}{\oplus}} L^2(\mathcal{R}_k) = L^2(\mathcal{R})$ with inner product

$$((u_k), (v_k))_{\mathcal{H}} = \sum_{i=1}^{N} (u_k, v_k)_{\mathcal{R}_k}.$$

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Strichartz estimates

we call a pair of exponents (p, q) **admissible** if

$$2 \leq p \leq +\infty, \quad \frac{2}{q} = \frac{1}{2} - \frac{1}{p}.$$

Then for any admissible exponents (p, q) and (p', q') we have

- The homogeneous estimate

$$\left\| e^{it\triangle} u_0 \right\|_{L^q(L^p(\mathcal{R}))} \leq c \|u_0\|_{L^2(\mathcal{R})}.$$

- The non-homogeneous estimate

$$\left\| \int_0^t e^{i(t-s)\triangle} h(.,s) ds \right\|_{L^q L^p(\mathbb{R}\times\mathcal{R})} \leq \|h\|_{L^{q'} L^{p'}(\mathbb{R}\times\mathcal{R})},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Local well-posedness

Theorem 1 (Ammari-Bchatnia-M)

Given $\varphi = (\varphi_1, ..., \varphi_N) \in L^2(\mathcal{R})$ and $\alpha \in (1, 5]$, then there exist $T = T(\varphi, \|a\|_{L^\infty}, \alpha, \lambda) > 0$ and a unique solution $u = (u_1, ..., u_N)$ of system (E) such that :

$$u_i \in \mathcal{C}([0, T]; L^2(\mathcal{R}_i)) \cap L^r\big((0, T); L^{\alpha+1}(\mathcal{R}_i)\big), 1 \leq i \leq N,$$

where $r = \frac{4(\alpha+1)}{\alpha-1}$. Moreover, the solution depends continuously on the initial data.

Sketch of the proof of Theorem 1

- Step 1 : Mild formulation of the solution $\rightarrow$ fixed point problem
- Step 2 : Strichartz estimate $\rightarrow$ contraction on a space-time ball $\rightarrow$ local in time existence and uniqueness.

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Global well-posedness

- Subcritical case : time of existence depends only on the L^2 norm of the data.

  $\int_0^{+\infty} |u|^2 \leq \|\varphi\|^2_{L^2(\mathcal{R})} \rightarrow$ iteration $\rightarrow$ global (in time) well-posedness.

- Critical case : time of depends also on the profile of the data.

  $\int_0^{+\infty} |u|^2 \leq \|\varphi\|^2_{L^2(\mathcal{R})}$ + Small data $\rightarrow$ iteration $\rightarrow$ global (in time) well-posedness.

Outline

Main results

Theorem 2 (Ammari-Bchatnia-M)

Consider $\alpha \in \{3,5\}$, a function a satisfying assumption (H) and the initial data φ in $L^2(\mathcal{R})$. For any solution u of the system (E), there exist $c > 0$ and $\omega > 0$ such that :

$$E_u(t) \leq c\, e^{-\omega t}, \quad \text{for all} \;\; t \geq 0,$$

provided the initial data satisfies $\|\varphi\|_{L^2(\mathcal{R})} \ll 1$, if one considers the case $\alpha = 5$.

Lemma 3

Consider $\alpha = \{3,5\}$. Let u be a solution associated to the system (E) with initial data $\varphi = (\varphi_1, \varphi_2, ..., \varphi_N) \in L^2(\mathcal{R})$ satisfying $\|\varphi\|_{L^2(\mathcal{R})} \ll 1$ for the case $\alpha = 5$. Then, for all $T \gg 1$, there exists a positive constant c which depends on T such that the following inequality holds,

$$\int_0^T \int_0^{+\infty} a(x)|u_1(s,x)|^2 dxds \geq c \left(\int_0^T \sum_{i=2}^{N} E_{u_i}(s)ds + \int_0^T \int_0^R |u_1(s,x)|^2 dxds \right).$$

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Essentials tools

Theorem 4

Consider $T > 0$, $\lambda = \pm 1$ and α an odd positive number. Let v be the global solution in $C\left([0,T]\,; L^2(\mathcal{R})\right) \cap L^2\left((0,T); L^\infty(\mathcal{R})\right)$ of

$$\begin{cases} i\partial_t v + \partial_x^2 v + \lambda |v|^{\alpha-1} v = 0, & (0,T)\times\mathbb{R}, \\ v(x,0) = v_0, & x \in \mathbb{R}. \end{cases}$$

Then, $v \in C^\infty([0,T]\times\mathbb{R})$ for all $v_0 \in L^2(\mathbb{R})$ with a compact support.

Theorem 5

Let $v \in C\left([0,T]; H^k(\mathbb{R})\right)$, be a strong solution of the equation

$$\begin{cases} i\partial_t v + \partial_x^2 v + \lambda |v|^{\alpha-1} v = 0, & (0,T)\times\mathbb{R}, \\ v(x,0) = v_0, & x \in \mathbb{R}. \end{cases}$$

If there is $t_1, t_2 \in [0,T]$, $t_1 \neq t_2$, $\rho > 2$ and $\beta > 0$ such that

$$v(.,t_1), v(.,t_2) \in H^1\left(e^{\beta|x|^\rho} dx\right),$$

then $v \equiv 0$.

Essentials tools

Lemma 6 (Lions'Lemma)

Let ω be an open bounded subset of $\mathbb{R} \times \mathbb{R}$. Consider $\{f_n\}_{n\in\mathbb{N}}$ a sequence in $L^q(\omega)$, $1 < q < \infty$, satisfying $\|f_n\|_{L^q(\omega)} \leq C$ and $f_n \longrightarrow f$ a.e. in ω. Thus $f_n \rightharpoonup f$ in $L^q(\omega)$, as $n \longrightarrow +\infty$.

Lemma 7 (Aubin-Lions'Lemma)

Let X_0, X, and X_1 be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. Suppose also that X_0 and X_1 are reflexive spaces. For $1 < p, q < \infty$, let $W = \left\{u \in L^p((0,T); X_0), \frac{du}{dt} \in L^q((0,T); X_1)\right\}$. Then the embedding of W into $L^p((0,T); X)$ is compact.

Sketch of the proof of the Lemma 3

By contradiction : we consider the case $\alpha = 3$.
Let $\{(u_i^k)(0)\}_{k\in\mathbb{N}}, 1 \leq i \leq N$ be a sequence of initial data attached with the solutions $\{u^k\}_{k\in\mathbb{N}} = \{(u_1^k, u_2^k, ..., u_N^k)\}_{k\in\mathbb{N}}$, which is assumed to be uniformly bounded by a constant $C > 0$, verify

$$\int_0^T \int_0^{+\infty} a(x)|u_1^k(s,x)|^2 dxds \leq \frac{1}{k}\left(\int_0^T \sum_{i=2}^{N} E_{u_i^k}(s)ds + \int_0^T \int_0^R |u_1^k(s,x)|^2 dxds\right)$$

- Step 1 :
  - $\lim_{k\to+\infty} \int_0^T \int_R^{+\infty} |u_1^k(s,x)|^2 dxds = 0.$
  - $u_1^k \to u_1$ strong in $L^2((0,T_1)\times(0,R))$
    $\Rightarrow u_1^k \to u_1$ a.e. in $[0,T_1]\times[0,R]$.
  - $$u_1^k \to \tilde{u}_1 = \begin{cases} u_1, & \text{a.e. in } [0,T_1]\times[0,R], \\ 0, & \text{a.e. in } [0,T_1]\times[R,+\infty[. \end{cases} \quad (1)$$

- Step 2 :
  First case : $u_1 \neq 0$
  - Our goal is to pass to the limit in :

$$i\partial_t u_1^k = -\partial_x^2 u_1^k - \lambda |u_1^k|^2 u_1^k - ia(x)u_1^k, \quad \text{in } \mathcal{D}'((0,T_1)\times\mathbb{R}_+).$$

  - $\{|u_1^k|^2 u_1^k\}_{k\in\mathbb{N}}$ is bounded in $L^{4/3}((0,T_1), L^4(\mathbb{R}_+))$ + Lions'lemma

$$\Downarrow$$

$$\begin{cases} i\partial_t u_1 + \partial_x^2 u_1 + \lambda |u_1|^2 u_1 = 0 & \text{a.e. in } [0,T_1]\times[0,R], \\ u_1 = 0, & \text{a.e. in } [0,T_1]\times[R,+\infty[. \end{cases} \quad (2)$$

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- $u_1 \in L^2((0,T_1);L^2(\mathbb{R}_+)), \exists\, t_0 \in (0,T_1)$ such that $u_1(t_0,.) \in L^2(\mathbb{R}_+)$ having compact support.

  Theorem 5 $\Downarrow$

  $u_1 \in \mathcal{C}^\infty((0,T_1)\times\mathbb{R}_+)$ with $u_1(t,x)=0$, for all $(t,x)\in(0,T_1)\times(R,+\infty)$.

  Theorem 6 $\Downarrow$

  $$u_1 \equiv 0 \text{ in } (0,T_1)\times\mathbb{R}_+.$$

- Conclude that $u_1 \equiv 0$ in $[0,T]\times\mathbb{R}_+$.

Contradiction

Second case : $u_1 \equiv 0$

- $\beta_k = \|u_1^k\|^2_{L^2((0,T)\times(0,R))} \underset{k\to+\infty}{\longrightarrow} 0$ and $v_1^k = \frac{u_1^k}{\beta_k}$ satisfies $\|v_1^k\|^2_{L^2((0,T)\times(0,R))} = 1$ and verifies :

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- follow the same steps as for the first case
  - $v_1^k \rightharpoonup \tilde{v}_1$ weakly in $L^2\Big((0,T_2), L^2(\mathbb{R}_+)\Big)$,
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$$\tilde{v}_1 = \begin{cases} v_1, & \text{a.e. in} \quad (0,T_2)\times[0,R], \\ 0, & \text{a.e. in} \quad (0,T_2)\times[R,+\infty[, \end{cases}$$

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- Holmogren's Theorem $\Rightarrow$ $v_1 \equiv 0$ in $(0,T_2)\times[0,R]$.
- Aubin-Lions's Lemma $\Rightarrow$
  $v_1^k \to 0$ strong in $L^2\Big((0,T)\times[0,R]\Big)$.

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Sketch of the proof of the Theorem 2.

- $\int_0^T E_u(s)ds \leq -\left(\frac{1}{2c} + \frac{1}{2\alpha_0}\right)\Big(E_u(T) - E_u(0)\Big).$
- $t \mapsto E_u(t)$ is a non-increasing function $\Rightarrow$ $E_u(T) \leq \frac{C(T)}{C(T)+T} E_u(0)$
- The semigroup property $\Rightarrow$ The exponential decay.

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Thanks for your attention ! ! !