Stabilization of Thermo-elastic transmission problem

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Tuniso-Libanese workshop in Control Theory and Related Fields

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Outline

Introduction

We consider a 1-D transmission problem between an elastic and thermoelastic equations, and there is a coupling with each one of these equation. Consider the following transmission problem.

$$\begin{cases} y_{tt} - y_{xx} + \beta(x) u_t = 0, & in \quad (0,l) \times \mathbb{R}^+, \\ u_{tt} - u_{xx} + \alpha\theta_x - \beta(x) y_t = 0, & \text{in} \quad (0,l) \times \mathbb{R}^+, \\ z_{tt} - z_{xx} + \beta(x) v_t = 0, & \text{in} \quad (l,L) \times \mathbb{R}^+, \\ v_{tt} - v_{xx} - \beta(x) z_t = 0, & \text{in} \quad (l,L) \times \mathbb{R}^+, \\ \theta_t - \theta_{xx} + \alpha u_{tx} = 0, & \text{in} \quad (0,l) \times \mathbb{R}^+, \end{cases} \tag{1}$$

where $\theta(x,t)$ is the temperature difference with respect to a fixed reference temperature, $u(x,t), y(x,t), z(x,t), v(x,t)$ denote the displacement of the system at time t in intervals $(0,l)$, $(0,l)$, (l,L) and (l,L) respectively, α is assumed to be positive constant, and the smooth function $\beta(x)$ such that $\beta(x) \geq \beta_0 > 0$.
The boundary conditions are as follows:

$$\begin{cases} y(0,t) = u(0,t) = 0, & t > 0, \\ z(L,t) = v(L,t) = 0, & t > 0, \end{cases} \tag{2}$$

and we have the following transmission conditions at $x = l$:

$$\begin{cases} y(l) = z(l), \quad u(l) = v(l), & t > 0, \\ y_x(l) = z_x(l), \quad u_x(l) = v_x(l), & t > 0, \\ \theta(0) = \theta(l) = 0 & \end{cases} \tag{3}$$

or the following transmission conditions at $x = l$:

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with the following initial conditions:

$$\begin{cases} y(x,0) = y_0, u(x,0) = u_0, y_t(x,0) = y_1, u_t(x,0) = u_1, \\ z(x,0) = z_0, v(x,0) = v_0, z_t(x,0) = z_1, v_t(x,0) = v_1, \\ \theta(x,0) = \theta_0. \end{cases} \quad (5)$$

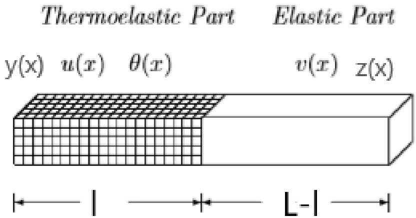

Figure: Transmission problem between elastic and thermo-elastic parts

Introduction

In 1960s, Dafermos in [2] considered the existence of the solution of the classical thermoelastic system and he showed the well posedness of the system and showed the asymptotic stability of the system under certain condition. Rivera further proved that the solution of this kind of thermoelastic system decays exponentially (see [5, 4]).

Introduction

In [3], they considered the asymptotic behavior of 1-dimensional bodies which are composed of two different types of materials, one of them is of thermoelastic type, while the other has no thermal effect. That is, he considered a material with a localized thermal effect. They proved the well posedness of the system and that the solution decays exponentially to zero as time goes to infinity.

Let $U = (y, y_t, u, u_t, z, z_t, v, v_t, \theta)$ be a regular solution of (1)-(5), its associated energy is defined by

$$E(t) = \frac{1}{2}\int_0^l (|y_t|^2 + |y_x|^2 + |u_t|^2 + |u_x|^2)dx \qquad (6)$$

$$+ \int_l^L (|z_t|^2 + |z_x|^2 + |v_t|^2 + |v_x|^2)dx + \frac{1}{2}\int_0^l |\theta|^2 dx.$$

A direct computation gives

$$\frac{d}{dt}E(t) = -\int_0^l |\theta_x|^2 dx \leq 0. \qquad (7)$$

Hence, the system (1) is dissipative in the sense that its energy is non-increasing.

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- **Logarithmic Stability:** $\exists\ c\ , \alpha > 0$ **such that**

$$E(t) \leq \frac{c}{(ln(1+t))^{\alpha}}||U_0||^2_{D(\mathcal{A})}, \quad \forall t > 0.$$

We reformulate the system (1) into an evolutionary problem in an appropriate Hilbert space.

Let the Hilbert energy space

$$\mathcal{H} = \left(H_L^1(0,l) \times L^2(0,l)\right)^2 \times \left(H_R^1(l,L) \times L^2(l,L)\right)^2 \times L^2(0,l),$$

where

$$H_L^1(0,l) = \{u \in H^1(0,l); u(0) = 0\}, H_R^1(l,L) = \{u \in H^1(l,L); u(L) = 0\}.$$

According to the thermoelastic transmission conditions we distinguish the following two unbounded linear operators $\mathcal{A}_1$ and $\mathcal{A}_2$:

$$\mathcal{A}_1 \begin{bmatrix} y \\ \tilde{y} \\ u \\ \tilde{u} \\ z \\ \tilde{z} \\ v \\ \tilde{v} \\ \theta \end{bmatrix} = \begin{pmatrix} \tilde{y} \\ y_{xx} - \beta(x)\tilde{u} \\ \tilde{u} \\ u_{xx} - \alpha\theta_x + \beta(x)\tilde{y} \\ \tilde{z} \\ z_{xx} - \beta(x)\tilde{v} \\ \tilde{v} \\ v_{xx} + \beta(x)\tilde{z} \\ \theta_{xx} - \alpha\tilde{u} \end{pmatrix}$$

with the domain

$$D(\mathcal{A}_1) = \left\{ (y, \tilde{y}, u, \tilde{u}, z, \tilde{z}, v, \tilde{v}, \theta) \in \mathcal{H}; \left| \begin{array}{l} y \in H^2(0,l), \tilde{y} \in H^1_L(0,l) \\ u \in H^2(0,l), \tilde{u} \in H^1_L(0,l) \\ z \in H^2(l,L), \tilde{z} \in H^1_R(l,L) \\ v \in H^2(l,L), \tilde{v} \in H^1_R(l,L) \\ \theta \in H^2(0,l) \\ u(l) = v(l), u_x(l) = v_x(l) \\ y(l) = z(l), y_x(l) = z_x(l) \\ \theta(0) = 0, \theta(l) = 0 \end{array} \right. \right\},$$

and

$$\mathcal{A}_2 \begin{bmatrix} y \\ \tilde{y} \\ u \\ \tilde{u} \\ z \\ \tilde{z} \\ v \\ \tilde{v} \\ \theta \end{bmatrix} = \begin{pmatrix} \tilde{y} \\ y_{xx} - \beta(x)\tilde{u} \\ \tilde{u} \\ u_{xx} - \alpha\theta_x + \beta(x)\tilde{y} \\ \tilde{z} \\ z_{xx} - \beta(x)\tilde{v} \\ \tilde{v} \\ v_{xx} + \beta(x)\tilde{z} \\ \theta_{xx} - \alpha\tilde{u} \end{pmatrix}$$

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$$D(\mathcal{A}_2) = \left\{ (y,\tilde{y},u,\tilde{u},z,\tilde{z},v,\tilde{v},\theta) \in \mathcal{H}; \left| \begin{array}{l} y \in H^2(0,l), \tilde{y} \in H_L^1(0,l) \\ u \in H^2(0,l), \tilde{u} \in H_L^1(0,l) \\ z \in H^2(l,L), \tilde{z} \in H_R^1(l,L) \\ v \in H^2(l,L), \tilde{v} \in H_R^1(l,L) \\ \theta \in H^2(0,l) \\ u(l) = v(l), u_x(l) = v_x(l) \\ y(l) = z(l), y_x(l) = z_x(l) \\ \theta(0) = 0, \theta_x(l) = \alpha \tilde{u}(l) \end{array} \right. \right\}.$$

System (1) can be rewritten as an evolutionary equation in $\mathcal{H}$:

$$\begin{cases} U(t) = \mathcal{A}_j U(t), t > 0, j = 1, 2, \\ U(0) = U_0, \end{cases} \tag{8}$$

where

$$U(t) = (y, \tilde{y}, u, \tilde{u}, z, \tilde{z}, v, \tilde{v}, \theta).$$

It's easy to see that $\mathcal{A}$ generates a C_0-Semigroup of contractions in the energy space $\mathcal{H}$.

Remark

We note that, when writing the operator $\mathcal{A}$ and $U \in D(\mathcal{A})$ this means that the result is true for $\mathcal{A}_{j=1,2}$ and $U \in D(\mathcal{A}_{j=1,2})$.

Strong Stability

Theorem(Arendt-Batty [1])

Assume that $\mathcal{A}$ is the generator of a C_0-semigroup of contractions $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If

1. $\mathcal{A}$ has no pure imaginary eigenvalues,
2. $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable,

where $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$, then the C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable.

Strong Stability

The resolvent $(I - \mathcal{A})^{-1}$ of $\mathcal{A}$ is compact, then $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$. Thus, the state of Arendt-Batty Theorem (see [1]) lessens to $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \Phi$.

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Theorem (Strong Stability)

The C_0-semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable on $\mathcal{H}$ in the sense that $\lim_{t\to+\infty}\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}}=0$ for all $U_0\in\mathcal{H}$.

Exponential Stability

Theorem(Exponential Stability)

There exists a constant $C > 0$, $w > 0$ such that for all initial data $U_0 \in D(\mathcal{A})$ the energy of the system (1)-(5) satisfies the following decay rate:

$$E(t) \leq Ce^{-wt}\|U_0\|^2_{D(\mathcal{A})}, \qquad \forall t > 0. \tag{9}$$

Sketch of the proof

Using the result of Huang-Pruss Theorem, (9) holds if and only if:

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{H1}$$

$$\sup_{\beta \in \mathbb{R}} \|(i\beta I - \mathcal{A})^{-1}\|_{L(\mathcal{H})} < +\infty. \tag{H2}$$

Sketch of the proof

Condition $(H1)$ can be derived from the result of strong stability. Suppose that $(H2)$ is false. Then, there exists $\beta_n \in \mathbb{R}$ and a sequence $U_n = (y_n, \tilde{y}_n, u_n, \tilde{u}_n, z_n, \tilde{z}_n, v_n, \tilde{v}_n, \theta_n) \in D(\mathcal{A})$ such that

$$|\beta_n| \to +\infty, \quad ||U_n||_{\mathcal{H}} = 1, \tag{10}$$

$$\lim_{n\to\infty} \|(i\beta_n I - \mathcal{A})U_n\|_{\mathcal{H}} = 0. \tag{11}$$

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We need to show that $||U_n||_H \to 0$ and hence we get a contradiction to (10).

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Thank you for your Attention!