STABILIZATION OF A GENERALIZED TELEGRAPH EQUATION ON STAR-SHAPED NETWORK

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30/10/2019

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In [1], S. Nicaise considered the stabilization of the generalized telegraph equation set in a real interval:

$$\begin{cases} V_t + gV + aI_x + kW &= 0, \text{ in } (0,L)\times(0,\infty), \\ I_t + rI + bV_x &= 0, \text{ in } (0,L)\times(0,\infty), \\ W_t + cW &= V, \text{ in } (0,L)\times(0,\infty), \end{cases} \tag{1}$$

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where, a, b, c, r, k and g are all non-negative functions in $L^\infty(0, L)$ that verify some assumptions mentioned in [1].

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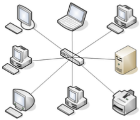
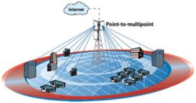
- Otherwise, a polynomial energy decay rate of type $\frac{1}{t}$ for smooth initial data is established.

Moreover, the obtained polynomial decay rate is optimal in the case $r = g = 0$.

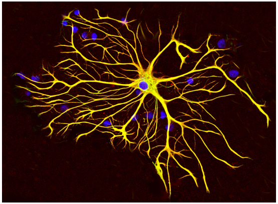
S. Nicaise. Stabilization and asymptotic behavior of a generalized telegraph equation. Z. Angew. Math. Phys., 66(6):3221–3247, 2015.

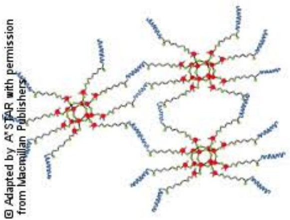
Applications of Networks



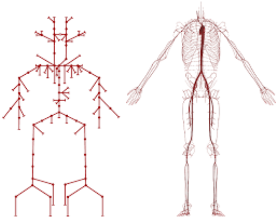


(A) Electrical Networks





(B) Chemical Networks



(C) Biological Networks

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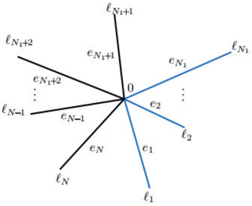
Objective



- Objective: To study the stabilization of the generalized telegraph equation on a one dimensional star shaped graph.

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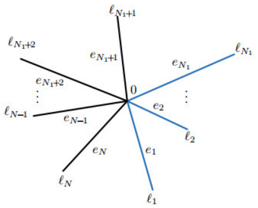

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- Main Result: The energy of the system decays exponentially to zero.

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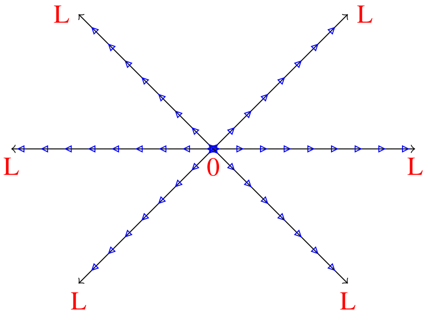

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We consider the following generalized telegraph equation on a one-dimensional star shaped network Γ, as follows

$$\begin{cases} V_{\ell,t} + g_\ell V_\ell + a_\ell I_{\ell,x} + k_\ell W_\ell & = 0, \text{ in } (0,L) \times \mathbb{R}_+, \ \ell \in \{0, ..., N\}, \\ I_{\ell,t} + r_\ell I_\ell + b_\ell V_{\ell,x} & = 0, \text{ in } (0,L) \times \mathbb{R}_+, \ \ell \in \{0, ..., N\}, \\ W_{\ell,t} + c_\ell W_\ell & = V_\ell, \text{in } (0,L) \times \mathbb{R}_+, \ \ell \in \{0, ..., N\}. \end{cases} \quad (2)$$

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All coefficients involved in (2) are in $L^\infty(\Omega)$, real valued and non negative. Moreover, we suppose that

$$a_\ell \geq \mu,\ b_\ell \geq \mu,\ c_\ell \geq \mu,\ k_\ell + g_\ell \geq \mu \quad a.e \text{ in } \Omega, \quad \ell = 0, .., N, \tag{3}$$

where μ is a positive real number.

Boundary Conditions

System (2) is subjected to the dissipative boundary condition on all the external vertices

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$$\begin{cases} \sum_{k=1}^{N} Z_{\ell k} I_{k,t}(0) & = \frac{1}{\delta}(V_0(0) - V_\ell(0)), \quad \forall \ell \in \{1, ..., N\}, \\ \\ V_{0,t}(0) & = \frac{-1}{\delta Y} \sum_{\ell=0}^{N} I_\ell(0), \end{cases} \tag{5}$$

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where, $Y > 0$, $\delta > 0$, and $Z = (Z_{\ell k})_{N \times N}$ is a symmetric, positive definite matrix.

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$$\langle (V, I, W, \nu, \eta)^\top, (V^*, I^*, W^*, \nu^*, \eta^*)^\top \rangle_{\mathcal{H}} = \sum_{\ell=0}^N \int_0^L (\theta_\ell V_\ell \overline{V_\ell^*} + \beta_\ell I_\ell \overline{I_\ell^*} + \gamma_\ell W_\ell \overline{W_\ell^*})\, dx$$
$$+ \delta Y\, \nu \overline{\nu^*} + \delta\, (\eta, (Z)^{-1} \eta^*)_{\mathbb{C}^N}, \quad \forall\, (V, I, W, \nu, \eta)^\top, (V^*, I^*, W^*, \nu^*, \eta^*)^\top \in \mathcal{H}. \tag{6}$$

Where

$$\theta = (\theta_\ell)_{\ell=0}^N,\ \beta = (\beta_\ell)_{\ell=0}^N,\ \gamma = (\gamma_\ell)_{\ell=0}^N \in \prod_{\ell=0}^N L^\infty(0, L)$$

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such that

$$\theta_\ell \geq \mu_0, \quad \beta_\ell \geq \mu_0, \quad \gamma_\ell \geq \mu_0 \quad \text{a.e in } \Omega, \quad \ell = 0, .., N, \tag{7}$$

for some constant $\mu_0 > 0$.

The Operator $\mathcal{A}$

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V_\ell(L) = \alpha_\ell I_\ell(L), \nu = V_0(0), \text{ and } \eta = (\eta_\ell)_{\ell=1}^{N} \text{ with } \eta_\ell = \sum_{k=1}^{N} Z_{\ell k} I_k(0)\}.
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and

$$\mathcal{A}(\mathbf{V}, \mathbf{I}, \mathbf{W}, \nu, \boldsymbol{\eta})^\perp = - \begin{pmatrix} (g_\ell V_\ell + a_\ell I_{\ell,x} + k_\ell W_\ell)_{\ell=0}^{N} \\ (r_\ell I_\ell + b_\ell V_{\ell,x})_{\ell=0}^{N} \\ (c_\ell W_\ell - V_\ell)_{\ell=0}^{N} \\ \frac{1}{\delta Y} \sum_{\ell=0}^{N} I_\ell(0) \\ \frac{1}{\delta}(V_\ell(0) - \nu)_{\ell=1}^{N} \end{pmatrix}.$$

Well posedness

If U is a sufficiently smooth solution of (2), then $U = (V, I, W, \nu, \eta) \in \mathcal{H}$ satisfies the first order evolution equation

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THEOREM

There exists $\theta,\ \beta,\ \gamma \in \prod_{\ell=0}^{N} L^\infty(0, L)$ satisfying (7) such that the operator $\mathcal{A}$ generates a C_0 semi-group of contractions on $\mathcal{H}$.

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$$\Re\langle \mathcal{A}U, U\rangle \leq -\frac{1}{2}\sum_{\ell=0}^{N}\int_{\Omega}(\theta_\ell g_\ell |V_\ell|^2 + 2\beta_\ell r_\ell |I_\ell|^2 + \gamma_\ell c_\ell |W_\ell|^2)dx - \sum_{\ell=0}^{N}\alpha_\ell |I_\ell(L)|^2 \leq 0, \quad (10)$$

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and

$$\lambda - \mathcal{A} \text{ is surjective for some } \lambda > 0.$$

This implies that (2) is well-posed and its solution admits the following representation,

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Theorem

For any initial data $U_0 \in \mathcal{H}$, problem (9) admits a unique weak solution $U \in C^0([0,\infty); \mathcal{H})$. Moreover if $U_0 \in D(\mathcal{A})$, problem (9) admits a unique strong solution $U \in C^1([0,\infty); \mathcal{H}) \cap C^0([0,\infty); D(\mathcal{A}))$.

Strong Stability

Theorem (Hayek-Nicaise-Salloum-Wehbe)

The C_0- *semi group* $(e^{t\mathcal{A}})_{t\geq 0}$ *is strongly stable on the energy space* $\mathcal{H}$*, i.e, for any* $U_0 \in \mathcal{H}$*, we have*

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The proof of the above theorem is based on the following lemmas :

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LEMMA (ARENDT-BATTY)

Let X be a reflexive Banach space and $(T(t))_{t\geq 0}$ be a bounded C_0 semi-group generated by A on X . If A has no eigenvalues on the imaginary axis and $\sigma(A) \cap i\mathbb{R}$ is countable, where $\sigma(A)$ denotes the spectrum of $\mathcal{A}$. Then, $(T(t))_{t\geq 0}$ is stable.

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Arendt -Batty theorem implies that $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable.

Asymptotic Behavior of the eigenvalues of $\mathcal{A}$ for $N = 1$ (2 cables)

THEOREM (HAYEK-NICAISE-SALLOUM-WEHBE)

In addition to (3), assume that the functions a, b, c, k are constant vectors and that $r_\ell = g_\ell = 0, \forall\, \ell$. Then, there exists $n_0 \in \mathbb{N}^*$ such that the sequences of eigenvalues of $\mathcal{A}, (\lambda_n)_{n \geq n_0}$ and $(\lambda_m)_{m \geq n_0}$ satisfy the following asymptotic behavior:

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$$\lambda_n = \begin{cases} \frac{\sqrt{a_0 b_0}}{2L} \log\left(\frac{1-\frac{\alpha_0\sqrt{b_0}}{\sqrt{a_0}}}{1+\frac{\alpha_0\sqrt{b_0}}{\sqrt{a_0}}}\right) + \frac{in\pi\sqrt{a_0 b_0}}{L} + o(1) & \text{if } \frac{\alpha_0\sqrt{b_0}}{\sqrt{a_0}} < 1, \\ \text{or} & \\ \frac{\sqrt{a_0 b_0}}{2L} \log\left(\frac{\frac{\alpha_0\sqrt{b_0}}{\sqrt{a_0}}-1}{\frac{\alpha_0\sqrt{b_0}}{\sqrt{a_0}}+1}\right) + \frac{i(n+\frac{1}{2})\pi\sqrt{a_0 b_0}}{L} + o(1) & \text{if } \frac{\alpha_0\sqrt{b_0}}{\sqrt{a_0}} > 1, \end{cases} \tag{11}$$

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Exponential Energy decay for $N \geq 1$

We proved the exponential stability of our system for all $N \geq 1$, based on a frequency domain approach.

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$$||e^{t\mathcal{A}}U_0||_{\mathcal{H}} \leq Me^{-\epsilon t}||U_0||_{\mathcal{H}}, \quad t \geq 0. \tag{13}$$

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$$(H2)\ \sup_{\lambda\in\mathbb{R}} ||(i\lambda - \mathcal{A})^{-1}||_{L(\mathcal{H})} = O(1),$$

where, $\rho(\mathcal{A})$ denotes the resolvent of the operator $\mathcal{A}$.

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- Generalization of the work to a tree-shaped network

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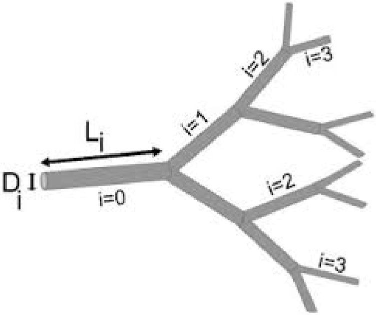
- Later on, we can consider the generalized telegraph equation in an arbitrary graphs with general transmission conditions at the interior nodes.



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4 References

References

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THANK YOU FOR YOUR ATTENTION