Finite Difference Scheme for 2D Parabolic Problem Modelling Electrostatic Micro-ElectroMechanical Systems

Hawraa ALSAYED

EDST Lebanese University and SISMI Université de Poitiers

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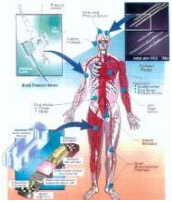
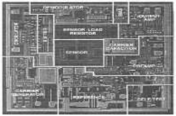
In collaboration with Hussein Fakih(Lebanese intenational University and Lebanese University), Alain MIRANVILLE (Université de Poitiers) and Ali WEHBE(Lebanese University)

Outline

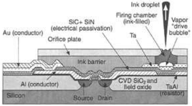
Introduction



Potential MEMS to monitor the condition of the body remotely.

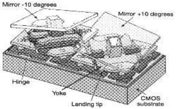


Annotated photomicrograph of an ADXL50 single-chip accelerometer.

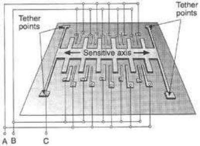


Cross-section of integrated thermal ink-jet chip.

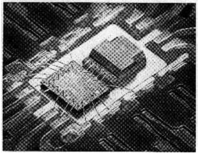
MEMS



Two pixels in the Texas instruments mirror array.



Sensing element of the ADXL50 accelerometer.



Motorola accelerometer chip and electronic chip.

Concepts for applications of automotive sensors and accelerometers.

The Problem

H. Alsayed, H. Fakih, A, Miranville, A. Wehbe, Electronic Research Announcements in Mathematical Sciences (2019)

The dynamic of the elastic dielectric membrane of an idealized MEMS is given by

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = \frac{\lambda f(x)}{(1-u)^2} \quad & \text{in } \Omega, \\ u(t,x) = 0 \quad & \text{on } \partial\Omega, \\ u(0,x) = 0 \quad & \text{in } \Omega. \end{aligned} \tag{1.1}$$

And the stationary state is given by

$$\begin{aligned} -\Delta u = \frac{\lambda f(x)}{(1-u)^2} \quad & \text{in } \Omega, \\ u(x) = 0 \quad & \text{on } \partial\Omega, \\ 0 \leq u < 1 \quad & \text{in } \Omega. \end{aligned} \tag{1.2}$$

Definition

The pull in voltage

$$\lambda^*(\Omega, f) = \sup\{\lambda > 0 | (1.2) \text{ possesses at least one solution}\}.$$

If $\lambda > \lambda^*$, then touchdown occurs.

So the existence of steady state is guaranteed upon finding an upper bound on the voltage λ, namely the pull in voltage.

Previous Works

P. Esposito, N. Ghoussoub, Y. Guo (2007)

Monotonicity of pull-in voltage
If $\Omega_1 \subset \Omega_2$, and f is a function satisfying $f \in C^{\beta}(\overline{\Omega}), \beta \in (0,1]; 0 \leq f \leq 1, f \not\equiv 0$ on Ω_2, then $\lambda^*(\Omega_1, f) \geq \lambda^*(\Omega_2, f)$ and the corresponding minimal solutions satisfy $u_{\Omega_1}(\lambda, x) \leq u_{\Omega_2}(\lambda, x)$ on Ω_1, $\forall 0 < \lambda < \lambda^*(\Omega_2, f)$.

If f_1, f_2 are two functions satisfying same conditions as above such that $f_1(x) \leq f_2(x)$ on Ω, then $\lambda^*(\Omega, f_1) \geq \lambda^*(\Omega, f_2)$ and we have $u_1(\lambda, x) \leq u_2(\lambda, x)$ on Ω, moreover if $f_1(x) \not\equiv f_2(x)$, then $u_1(\lambda, x) < u_2(\lambda, x)$.

Spectral properties of minimal solutions
Let $u, v \in H_0^1(\Omega)$- weak solution, super solution of the stationary problem respectively. If $\mu_1(\lambda, u) > 0$, then $u \leq v$ a.e. in Ω. If $\mu_1(\lambda, u) = 0$, then $u = v$ a.e. in Ω.

Estimates for the pull-in voltage. Assume that f satisfies $f \in C^{\beta}(\overline{\Omega}), \beta \in (0,1]; 0 \leq f \leq 1, f \neq 0$ on a bounded domain Ω in $\mathbb{R}^n$, then $\exists$ a finite pull-in voltage $\lambda^* := \lambda^*(\Omega, f) > 0$ such that

1) If $0 \leq \lambda < \lambda^*$; $\exists$ at least one solution for the stationary problem.
2) If $\lambda > \lambda^*$; there is no solution for the stationary problem.
3) The following bounds on λ^* hold for any bounded domain Ω:

$$\underline{\lambda} := \max\left\{\frac{8N}{27}, \frac{2(3N-4)}{9}\right\} \frac{1}{\sup_{\Omega} f} \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \leq \lambda^*(\Omega, f).$$

$$\min\left\{\overline{\lambda_1} := \frac{4\mu_\Omega}{27 \inf_{x\in\Omega} f(x)}, \overline{\lambda_2} := \frac{\mu_\Omega \int_\Omega \phi_\Omega}{3\int_\Omega f\phi_\Omega dx}\right\} \geq \lambda^*(\Omega, f)$$

Global convergence when $\lambda < \lambda^*$

For $\lambda < \lambda^* := \lambda^*(\Omega, f)$ there exits a unique global solution $u(x,t)$ for the dynamical system which monotonically converges as $t \to +\infty$ to the unique minimal solution $u_\lambda(x)$ of the stationary problem.

Touch down when $\lambda > \lambda^*$

For $\lambda > \lambda^*$ there exists a finite time $T_\lambda(\Omega, f)$ at which the unique solution $u(x,t)$ of the dynamical system must touch down. Moreover, if $\inf_{x\in\Omega} f(x) > 0$, then we have the bound

$$T_\lambda(\Omega, f) \le T_{0,\lambda} := \frac{8\,(\lambda+\lambda^*)^2}{3\inf_{x\in\Omega} f(x)\,(\lambda-\lambda^*)^2\,(\lambda+3\lambda^*)}\left[1+\left(\frac{\lambda+3\lambda^*}{2\lambda+2\lambda^*}\right)^2\right]$$

Estimates for finite touchdown times

Suppose u_1 respectively u_2 associated to λ and permittivity profiles f_1, f_2 with corresponding touchdown time $T_\lambda(\Omega, f_1)$, $T_\lambda(\Omega, f_2)$. If $f_1(x) \ge f_2(x)$ on Ω and if $f_1(x) > f_2(x)$ then necessarily $T_\lambda(\Omega, f_1) < T_\lambda(\Omega, f_2)$.
As well if u_1, respectively u_2 associated to λ_1, λ_2 and which has a finite touchdown time $T_{\lambda_1}(\Omega, f)$, respectively $T_{\lambda_2}(\Omega, f)$. If $\lambda_1 > \lambda_2$, then necessarily $T_{\lambda_1}(\Omega, f) < T_{\lambda_2}(\Omega, f)$.

L. CHERFILS, A. MIRANVILLE, S. PENG, C. XU, Electronic Research Announcements in Mathematical Sciences (2018)

Assuming that u_λ is the unique minimal solution.

The semi-implicit and implicit semi-discrete scheme

Assuming that $0 \leq u_0(x) \leq u_\lambda(x)$ a.e. $x \in \Omega$, there holds $\forall n \in \mathbb{N} \cup \{0\}$ a.e. $x \in \Omega$ in particular $0 \leq u_n(x) < 1$ a.e. $x \in \Omega$.

As well for $0 \leq u_n \leq u_\lambda < 1$ a.e. $x \in \Omega$, then the implicit semi-discrete scheme possesses at least one solution such that $0 \leq u_{n+1} \leq u_\lambda < 1$ a.e. $x \in \Omega$.

It was shown also that $\forall n \in \mathbb{N} \cup \{0\}$,

$$\left\| u_{n+1} - u_\lambda \right\| \leq \frac{1}{1 + c_0 \tau} \left(1 + \frac{2\lambda\tau}{(1 - \overline{u})^3} \right) \left\| u^n - u_\lambda \right\|,$$

therefore u_n converges to u_λ if $\lambda < \dfrac{1}{2} c_0 \left(1 - \overline{u}\right)^3$ for both schemes.

The stationary problem admits a solution if $0 \leq \lambda \leq \dfrac{8(1-\delta)\delta^2}{(b-a)}$.

If $\lambda < \dfrac{\left(1 - \|U^*\|_\infty\right)^3}{(b-a)^2}$, then $\{U^n\}$ converges to U^*.

Our Results

For $n \in \mathbb{N} \cup \{0\}$, $1 \leq i, j \leq M$, the fully discretized semi implicit 2D scheme is written as follows:

$$u_{i,j}^{n+1}\left(1+\frac{4\tau}{h^2}\right) - \frac{\tau}{h^2}u_{i+1,j}^{n+1} - \frac{\tau}{h^2}u_{i-1,j}^{n+1} - \frac{\tau}{h^2}u_{i,j+1}^{n+1} - \frac{\tau}{h^2}u_{i,j-1}^{n+1} =$$

$$u_{i,j}^{n} + \frac{\tau\lambda f(x_i, y_j)}{(1-u_{i,j}^{n})^2}.$$

and which consequently can be written in the vector form

$$AU^{n+1} = G(U^n).$$

The matrix A is given by

$$A = diag(-C, B, -C),$$

where $B = diag(-\frac{\tau}{h^2}, 1 + \frac{4\tau}{h^2}, -\frac{\tau}{h^2})$ and $C = \frac{\tau}{h^2} I_{M\times M}$.

$$G(U^n) = \left(u^n_{1,1} + \frac{\lambda f(x_1, y_1)}{(1 - u^n_{1,1})^2}, \cdots, u^n_{M,M} + \frac{\lambda f(x_M, y_M)}{(1 - u^n_{M,M})^2} \right)^T.$$

It is easy to see that the matrix A is positive definite, invertible and therefore its inverse is positive definite as well.

This guarantees the existence of the discrete solution.

Existence of steady state

The stationary equation has the following form

$$LU^* = \lambda F(U^*) \tag{3.1}$$

where, $L = \frac{1}{h^2} diag(-I, B, -I)$ and $B = diag(-1, 4, -1)$.

$$F(U^*) = \left(\frac{f(x_1, y_1)}{(1 - u^*_{1,1})^2}, \cdots, \frac{f(x_M, y_M)}{(1 - u^*_{M,M})2} \right)^T.$$

Theorem

Assume that $\lambda \leq \frac{2\pi^2\delta^2(1-\delta)}{(b-a)^2}$, for some $\delta \in (0, 1)$. Then (3.1) admits a solution U^* such that $0 \leq U^* < E$, where $E = (1, 1, \cdots, 1)^t_{1\times M^2}$.

Proof.

- Write the problem as $U^* = \lambda L^{-1} F(U^*) = H(U^*)$.
- $\left\| L^{-1} \right\|_2 = \dfrac{1}{\mu_1} = \dfrac{(b-a)^2}{2\pi^2}$.
- Consider the sequence $W^{k+1} = H(W^k)$.
- $\{W^{k+1}\}$ is increasing, and is bounded from above iff

$$\lambda \leq \frac{2\pi^2\delta^2(1-\delta)}{(b-a)^2}.$$

□

Convergence

Theorem

The discrete solution $\{U^n\}_n$ *converges to the stationary solution* U^* *if* $\lambda < \dfrac{(1-\|U^*\|_\infty)^3}{(b-a)^2}$.

Proof.

- We set $V^{n+1} = U^{n+1} - U^*$.
- $AU^{n+1} = U^n + F(U^n)$.
- $V^{n+1} = A^{-1}(V^n + F(U^n - F(U^*)))$.
- $((AV^{n+1}, V^{n+1})) \leq \|V^n + F(U^n) - F(U^*)\| \left\|V^{n+1}\right\|$.
- $((AV^{n+1}, V^{n+1})) \geq \left(1 + \dfrac{2\tau}{(b-a)^2}\right) \left\|V^{n+1}\right\|^2$.
- $\left\|V^{n+1}\right\| \leq \left(1 + \dfrac{2\tau}{(b-a)^2}\right)^{-1} \left(1 + 2\lambda\tau \dfrac{1}{(1-\|U^*\|_\infty)^3}\right) \|V^n\|$.

$\{U^n\}$ converges to U^* if $\lambda < \dfrac{(1-\|U^*\|_\infty)^3}{(b-a)^2}$, □

2 D Implicit Scheme

We can write the implicit scheme as: $F(U^{n+1}) = 0$ where

$$F : [0,1[^{M^2} \longrightarrow \mathbb{R}^{M^2}$$

and

$$F_{i,j}(U^{n+1}) = \left(1 + 4\frac{\tau}{h^2}\right) u_{i,j}^{n+1} - u_{i,j}^{n} - \frac{\tau}{h^2}\left(u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}\right)$$
$$-\frac{\tau\lambda f}{(1 - u_{i,j}^{n+1})^2}, \quad i,j = 1,\cdots,M,$$

Using Newton Method,

$$U^{n+1} = -DF^{-1}(U^n)F(U^n) + U^n, \tag{4.1}$$

The Jacobian matrix is given by

$$\frac{\partial F_{i,j}}{\partial u_{r,s}} = \begin{cases} -\frac{\tau}{h^2} & \text{if} \quad r = i+1 \quad \text{or} \quad r = i-1 \quad \text{and } s = j \\ & \text{and if } s = j+1 \text{ or } s = j-1 \quad \text{and } r = i \\ 1 + 4\frac{\tau}{h^2} - \frac{2\lambda\tau f(x_i, y_j)}{(1 - u_{i,j}^{n+1})^3} & \text{if} \quad r = i \text{ and } s = j \end{cases}$$

Theorem

The Jacobian matrix is invertible if $\tau < \dfrac{(b-a)^2}{2\pi^2\delta^2(1-\delta)}$, *for some* $\delta \in (0,1)$

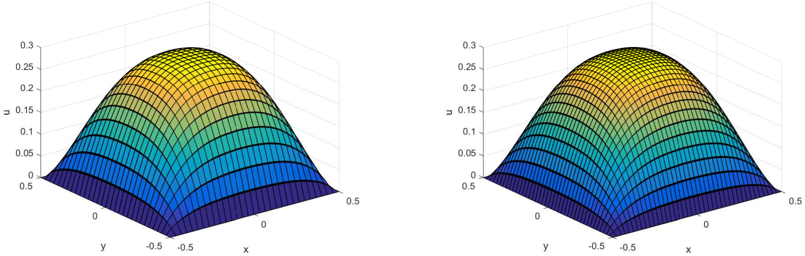
Proof.

- $DF(U^*) = \left(1 + \dfrac{4\tau}{h^2}\right) I - M,$
- $M = diag(\dfrac{\tau}{h^2} I, H_{i,j}, \dfrac{\tau}{h^2} I),\ j = 1, \cdots, M.$
- $H_{i,j} = diag(\dfrac{\tau}{h^2}, \dfrac{2f_{1,j}\lambda\tau}{(1-u^*_{1,j})^3}, \dfrac{\tau}{h^2}),\ \ i = 1, \cdots, M.$
- $\|M\|_\infty = \max\{\dfrac{2f_{i,j}\lambda\tau}{(1-u^*_{i,j})^3} + 2\dfrac{\tau}{h^2}, \dfrac{2f_{l,k}\lambda\tau}{(1-u^*_{l,k})^3} + 4\dfrac{\tau}{h^2}, \dfrac{2f_{s,t}\lambda\tau}{(1-u^*_{s,t})^3} + 3\dfrac{\tau}{h^2}\}.$

□

Numerical Simulations

Figure 1 shows for both schemes the behavior of the solution u at each time step with respect to the spatial variables x and y.



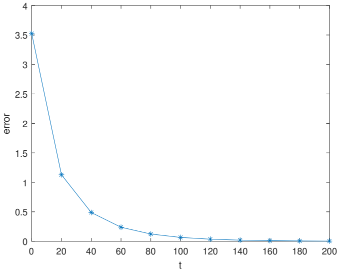
Semi-implicit scheme solution

Implicit scheme solution

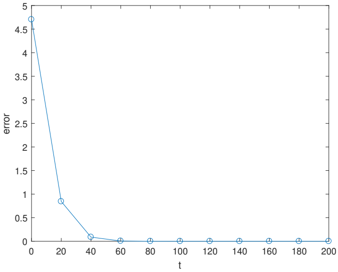
Figure: $\lambda = 10,\ f(x,y) = \sqrt{x^2 + y^2},\ \tau = 0.01,\ M = 29$

Errors

Figure 2 shows the behavior of the error, $\left\| u^{n+1} - u^{n} \right\|$, which converges to zero with respect to time.



Semi-implicit scheme error

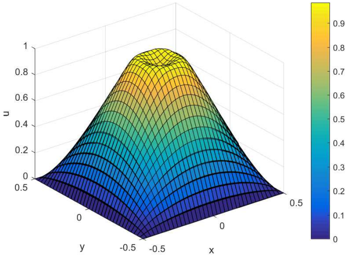


Implicit scheme error

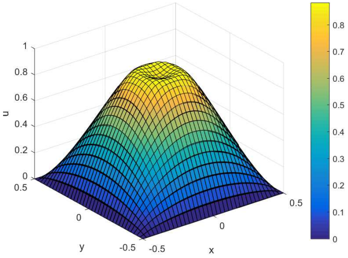
Figure: $\lambda = 10, \ f(x, y) = \sqrt{x^2 + y^2}, \ \tau = 20, \ M = 29$

touchdown

The touchdown phenomenon for both schemes is represented by the following figures.



Semi-implicit scheme touchdown



Implicit scheme touchdown

Figure: $\lambda = 11.5,\ f(x,y) = \sqrt{x^2+y^2},\ \tau = 0.001,\ M = 35$, in 3 touchdown is observed at $t = 682\tau$ however in 3 it is observed at $t = 1.3339\tau$

Thanks for your Attention!