Fixed point Theorems in metric Spaces via C-class function

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- In this paper, we present new type contractions involving C-class functions and establish several common fixed point theorems for this class of mappings
- Our results extend and complement some theorems given in the literature

Preliminaries

We recall some well known notions and definition of the *b*-metric spaces.

Definition

Let X be a nonempty set and $d : X \times X \longrightarrow [0, +\infty)$. A function d is called a *b*-metric with constant $s \geq 1$ if
b(0) $d(x, y) = 0$ if and only if $x = y$;
b(1) $d(x, y) = d(y, x)$ for all $x, y \in X$;
b(2) $d(x, y) \leq s[d(x, z) + d(z, y)]$, for all $x, y, z \in X$.
In this case, the pair (X, d) is called a *b*-metric space with coefficient s.

Preliminaries

It is obvious that a b-metric space with base $s = 1$ is a metric space then it is clear that definition of b-metric space is an extension of usual metric space.
In 2014, A.H.Ansari [1] introduced the concept of a C-class functions which covers a large class of contractive conditions

Definition

[1] A continuous function $F : [0, \infty)^2 \to \mathbb{R}$ is called C-class function if for any $s, t \in [0, \infty)^2$; the following conditions hold
$c1$ $F(s, t) \leq s$;
$c2$ $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.
An extra condition on F that $F(0, 0) = 0$ could be imposed in some cases if required. The letter C will denote the class of all C- functions.

Preliminaries

Preliminaries

The following examples shows that the class C is nonempty:

Example

1. $F(s,t) = s - t$:
2. $F(s,t) = ms$; for some $m \in (0,1)$.
3. $F(s,t) = \frac{s}{(1+t)^r}$ for some $r \in (0,1)$.

4.$F(s,t) = \frac{\log(t+a^s)}{(1+t)}$, for some $a > 1$.

Let u denote the class of the functions $\varphi : [0,1) \longrightarrow [0,1)$ which satisfy the following conditions:

a) φ is continuous ;

b) $\varphi(t) > 0,\ t > 0$ and $\varphi(0) \geq 0$.

Preliminaries

Definition

[2] A function $\psi : [0,1) \longrightarrow [0,1)$ is called an altering distance function if the following properties are satisfied:
i) ψ is non-decreasing and continuous,
ii) $\psi(t) = 0$ if and only if $t = 0$.

Let us suppose that Ψ denote the class of the altering distance functions.

Preliminaries

Definition

A tripled (ψ, φ, F) where $\psi \in \Psi$; $\varphi \in \Phi_u$ and $F \in C$ is said to be a monotone if for any $x, y \in [0,1)$;

$$x \leq y \Rightarrow F(\psi(x), \varphi(x)) \leq F(\psi(y), \varphi(y)).$$

Example

Let $F(s,t) = s - t$, $\varphi(x) = \sqrt{x}$

$$\psi(x) = \begin{cases} \sqrt{x} & if \quad 0 \leq x \leq 1 \\ x^2 & if \quad x > 1 \end{cases},$$

then (ψ, φ, F) is monotone.

Main Result

In this section, we give two fixed point theorems in metric spaces.

Main Result

Theorem

Let (X, d) be a $b-$complete metric space and let T be a self-mapping on X that satisfies the following contractive condition:

$$\psi\left(d(Tx, Ty)\right) \leq F\left(\psi(M(x,y)), \varphi(M(x,y))\right)$$

for all $x, y \in X$ .where $\psi \in \Psi$; $\varphi \in \Phi_u$ and $F \in C$ such that (ψ, φ, F) is monotone and

$$M(x,y) = \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{d(x,Ty) + d(y,Tx)}$$

Then T has a unique fixed point in X.

Main Result

Proof.

Let x in X and $\{x_n\}_n$ be a sequence in X defined as following

$$Tx_n = x_{n+1}, y_n = x_{n-1} \ n = 0, 1, 2 \ldots$$

$$\psi\left(d(Tx_n, Tx_{n-1})\right) \leq F\left(\psi(M(x_n, x_{n-1})), \varphi(M(x_n, x_{n-1}))\right)$$

Thus

$$\begin{aligned} \psi\left(d(Tx_n, Tx_{n-1})\right) &\leq F\left(\psi(d(x_n, x_{n-1})), \varphi(d(x_n, x_{n-1}))\right) \\ &\leq \psi(d(x_n, x_{n-1})) \end{aligned}$$

Since ψ is non-decreasing, then $d(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n-1})$. This means $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. □

Main Result

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Proof.

Thus it converges and there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Let $n \to \infty$, then contractive condition implies

$$\psi(r) \leq F\left(\psi(r), \lim_{n\to\infty} \inf \varphi(d(x_n, x_{n-1}))\right) \leq F(\psi(r), \varphi(r)) \leq \psi(r)$$

Thus $\psi(r) = 0$. Therefore $r = 0$, that is $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$
Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. □

Proof.

Then there exists an $\varepsilon > 0$ for which we can nd two sequences of positive integers $m(k)$ and $n(k)$ such
that for all positive integers k, $n(k) > m(k) > k$ and
$d(xm(k); xn(k)) \geq \varepsilon$
Let $n(k)$ be the smallest such positive integer $n(k) > m(k) > k$ such that

$$d(x_{m(k)}; x_{n(k)}) \geq \varepsilon; d(x_{m(k)}; x_{n(k)-1}) \leq \varepsilon$$

So wer find $\psi(\varepsilon) = 0$ which is a contradiction. Thus $\{x_n\}$ is a b-Cauchy sequence in X.. Since (X, d) is a complete b-metric space, there exists $u \in X$.
there exists$u \in X$ such that $\lim_{n\to\infty} x_n = u$ □

Proof.

Uniqueness of Fixed Point:

Moreover, u is a unique fixed point of T. Let $v \neq u$ be another fixed point of f.

From the contraction condition, we have

$$\psi\left(d(u,v)\right) \leq \psi\left(sd(u,v)\right) = \psi\left(sd(Tu,Tv)\right) \leq F\left(\psi(M(u,v)), \varphi(M(u,v))\right)$$

$$M(u,v) = \frac{d(u,Tu)d(u,Tv) + d(v,Tv)d(v,Tu)}{d(u,Tv) + d(y,Tu)}$$

Then $\psi\left(d(u,v)\right) \leq 0$ thus $d(u;v) = 0$ that is $u = v$. This shows T has a unique fixed point. □

- A. H.Ansari, Note on φ-ψ-contractive type mappings and related fixed point", The 2nd regional conference on mathematics and applications, Payame Noor University, 377-380, 2014.
- S. Banach, Sur les operations dans les ensembles abstraits et leur application aux équations int égrales, Fundam. Math., 3 (1922), 133-181.
- M. S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, Bulletin of the Australian Mathematical Society, 30,1, 1–9, 1984.
- I.A. Bakhtin, The contraction mapping principle in quasimetric spaces (Russian), Func. An., Gos. Ped. Inst. Unianowsk, 30 (1989),pp. 26-37.
- S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inf. Univ. Ostrav., 1 (1993), pp. 5-11..


Thanks!
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