Stability of Abstract System With Fractional Order Damping

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Outline

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Consider a linear evolution equation on Hilbert space $\mathcal{H}$

$$\begin{cases} u_t = \mathcal{B}u, \\ u(0) = u_0 \in \mathcal{H}. \end{cases}$$

Assume that $\mathcal{B}$ generates a C_0-semigroup of contractions $e^{t\mathcal{B}}$ in $\mathcal{H}$.

Definition

The C_0-semigroup of contractions $e^{t\mathcal{B}}$ is

1. exponentially stable if there are constants $M \geq 1, \ \epsilon > 0$ such that

$$\left\|e^{t\mathcal{B}}\right\| \leq Me^{-\epsilon t}, \ \ t \geq 0, \text{ i.e., } \left\|e^{t\mathcal{B}}u_0\right\|_{\mathcal{H}} \leq M\, e^{-\epsilon t}\|u_0\|_{\mathcal{H}}, \ \ \forall\, u_0 \in \mathcal{H}.$$

2. polynomially stable of order $\alpha > 0$ if there exists a positive constant C such that

$$t^{\frac{1}{\alpha}} \left\|e^{t\mathcal{B}}\mathcal{B}^{-1}\right\| = O(1), \text{ i.e., } \left\|e^{t\mathcal{B}}u_0\right\|_{\mathcal{H}} \leq \frac{C}{t^{\frac{1}{\alpha}}} \left\|u_0\right\|_{D(\mathcal{B})}, \ \forall\, u_0 \in D\left(\mathcal{B}\right).$$

Definition

The C_0-semigroup of contractions $e^{t\mathcal{B}}$ is

3. analytic if there exists a constant $C > 0$ such that
$$\left\|\mathcal{B}e^{t\mathcal{B}}\right\| \leq \frac{C}{t}, \quad t > 0.$$

4. of Gevrey class of order $\delta > 1$ if for any compact set $K \subset (0, \infty)$ and any $\tau > 0$, there exists a constant $C > 0$ such that
$$\left\|\mathcal{B}^n e^{t\mathcal{B}}\right\| \leq C\tau^n (n!)^\delta, \quad t \in K,\ n \geq 0.$$

Frequency Domain

Theorem. Assume that $i\mathbb{R} \subset \rho(\mathcal{B})$.

1. $e^{t\mathcal{B}}$ is analytic if and only if
$$\overline{\lim_{\lambda\to\infty}} |\lambda| \left\| (i\lambda - \mathcal{B})^{-1} \right\| < \infty.$$
2. $e^{t\mathcal{B}}$ is Gevrey class of order $\delta > 1$ if and only if
$$\overline{\lim_{\lambda\to\infty}} |\lambda|^{\frac{1}{\delta}} \left\| (i\lambda - \mathcal{B})^{-1} \right\| < \infty.$$
3. $e^{t\mathcal{B}}$ is exponentially stable if and only if
$$\overline{\lim_{\lambda\to\infty}} \left\| (i\lambda - \mathcal{B})^{-1} \right\| < \infty.$$
4. $e^{t\mathcal{B}}$ is polynomially stable of order $\alpha > 0$ if and only if
$$\overline{\lim_{\lambda\to\infty}} |\lambda|^{-\frac{1}{\alpha}} \left\| (i\lambda - \mathcal{B})^{-1} \right\| < \infty.$$

Theorem

Let $(\lambda_{k,n})_{1\leq k\leq K,\ n\geq 1}$ the k-th branch of **eigenvalues** of $\mathcal{B}$ and $\{e_{k,n}\}_{1\leq k\leq K,\ n\geq 1}$ the system of **eigenvectors** which forms a **Riesz basis** in $\mathcal{H}$. Then, $e^{t\mathcal{B}}$ is **exponentially stable** if and only if

$$S(\mathcal{B}) := \sup\left\{\Re\left(\lambda\right) :\ \lambda\in\sigma\left(\mathcal{B}\right)\right\} < 0.$$

Theorem Loreti and Rao in [5] (2006)

Let $(\lambda_{k,n})_{1\leq k\leq K,\ n\geq 1}$ denotes the k-th branch of **eigenvalues** of $\mathcal{B}$ and $\{e_{k,n}\}_{1\leq k\leq K,\ n\geq 1}$ the system of **eigenvectors** which forms a **Riesz basis** in $\mathcal{H}$. Assume that for each $1\leq k\leq K$ there exists a positive sequence $(\mu_{k,n})_{1\leq k\leq K,n\geq 1}$; $\mu_{k,n}\to+\infty$ and two positive constants $\alpha_k\geq 0$, $\beta_k>0$ such that

$$\Re(\lambda_{k,n})\approx-\frac{\beta_k}{\mu_{k,n}^{\alpha_k}}\quad\text{and}\quad|\mathrm{Im}(\lambda_{k,n})|\approx\mu_{k,n}\quad\forall n\geq 1.$$

Then

$$E(t):=\frac{1}{2}\|e^{t\mathcal{A}}u_0\|_{\mathcal{H}}^2\leq\frac{M\,\|u_0\|_{D(\mathcal{B})}^2}{t^{2\delta}}\quad\forall u_0\in D(\mathcal{B}),\ t>0,$$

$$\delta:=\min_{1\leq k\leq K}\frac{1}{\alpha_k}.$$

Example (Liu and Zheng [4])

Let A is a self-adjoint coercive operator with compact resolvent in a separable Hilbert space X.

$$\begin{cases} u_{tt} + Au + A^{\gamma} u_t = 0, \\ u|_{t=0} = u_0, \; u_t|_{t=0} = u_1, \end{cases} \tag{1}$$

where $0 \leq \gamma \leq 1$.

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1. $\gamma = 0$ viscous damping.
2. $\gamma = \frac{1}{2}$ square-root damping.
3. $\gamma = 1$ Kelvin-Voigt damping.

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3. $\gamma = 1$ Kelvin-Voigt damping.

Let

$$\mathcal{H} = D\left(A^{\frac{1}{2}}\right) \times X.$$

If we introduce $v = u_t$, then system (1) becomes

$$\begin{cases} y_t = \mathcal{B}_\gamma y, \\ y|_{t=0} = y_0 := (u_0, u_1), \end{cases} \tag{2}$$

with $y = (u, v)$, $\mathcal{B}_\gamma = \left(v, -A^{\frac{1}{2}}\left(A^{\frac{1}{2}}u + A^{\gamma-\frac{1}{2}}v\right)\right)$ and

$$D(\mathcal{B}_\gamma) = \left\{(u, v) \in D\left(A^{\frac{1}{2}}\right) \times D\left(A^{\frac{1}{2}}\right) \text{ and } A^{\frac{1}{2}}u + A^{\gamma-\frac{1}{2}}v \in D\left(A^{\frac{1}{2}}\right)\right\}.$$

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Moreover,

$$E(t) = \frac{1}{2}\|(u, u_t)\|^2_{\mathcal{H}}$$

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Moreover,

$$E(t) = \frac{1}{2}\|(u, u_t)\|_{\mathcal{H}}^2 \implies E'(t) = -\left\|A^{\frac{\gamma}{2}}u_t\right\|_X^2 \leq 0.$$

Theorem

The operator $\mathcal{B}_\gamma$ generates a C_0-semigroup of contractions $e^{t\mathcal{B}_\gamma}$ in $\mathcal{H}$.

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1. For $\frac{1}{2} \leq \gamma \leq 1$ the semigroup of contractions $e^{t\mathcal{B}_\gamma}$ is exponentially stable and analytic.
2. For $0 \leq \gamma < \frac{1}{2}$ the semigroup of contractions $e^{t\mathcal{B}_\gamma}$ is exponentially stable. While, for $0 < \gamma < \frac{1}{2}$ the semigroup of contractions $e^{t\mathcal{B}_\gamma}$ is in Gevrey's class.

$$\begin{cases} u_{tt} + Au + Bu_t = 0, \\ u|_{t=0} = u_0, \ u_t|_{t=0} = u_1, \end{cases} \tag{3}$$

where B is a self-adjoint operator, such that $\rho_1 A^\gamma \leq B \leq \rho_2 A^\gamma$ $(0 < \rho_1 < \rho_2 < \infty)$.

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Theorem

1. For $\frac{1}{2} \leq \gamma \leq 1$; Chen and Triggiani in [1] (1989) showed that the semigroup corresponding to (3) is analytical.
2. For $0 < \gamma < \frac{1}{2}$; Chen and Triggiani in [2] (1990) showed that the semigroup corresponding to (3) is in Gevrey's class.
3. For $\gamma < 0$; Liu and Zhang in [3] (2015) showed that the semigroup corresponding to (3) is polynomially stable of order $\frac{1}{2|\gamma|}$.

Loreti and Rao in [5] (2006)

$$\begin{cases} u_{tt} + Au + A^{\gamma} u_t + \alpha y = 0, \\ y_{tt} + Ay + \alpha u = 0, \\ u|_{t=0} = u_0, \; u_t|_{t=0} = u_1, \; y|_{t=0} = y_0, \; y_t|_{t=0} = y_1, \end{cases} \tag{4}$$

where $\gamma < 0$, $\alpha \in \mathbb{R}^*$ and A is a self-adjoint, coercive operator with a compact resolvent in a separable Hilbert space X.

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where $\gamma < 0$, $\alpha \in \mathbb{R}^*$ and A is a self-adjoint, coercive operator with a compact resolvent in a separable Hilbert space X.

1. System (4) is not exponentially stable.
2. An optimal polynomial energy decay rate of type $t^{-\tau(\gamma)}$ is obtained, such that

$$\tau(\gamma) = \begin{cases} \frac{1}{\gamma+1}, & -\dfrac{1}{2} \leq \gamma \leq 0, \\ -\frac{1}{\gamma}, & -\dfrac{1}{2} \geq \gamma. \end{cases}$$

We consider following abstract system

$$\begin{cases} u_{tt} + aAu + A^{\gamma} u_t + \alpha y_t = 0, \\ y_{tt} + Ay - \alpha u_t = 0, \\ u|_{t=0} = u_0,\ u_t|_{t=0} = u_1,\ y|_{t=0} = y_0,\ y_t|_{t=0} = y_1, \end{cases} \tag{5}$$

where $a > 0$, $\gamma \leq 0$, $\alpha \in \mathbb{R}^*$ is the coupling parameter and A is a self-adjoint, coercive operator with a compact resolvent in a separable Hilbert space X.

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Let

$$\mathcal{H} = D(A^{\frac{1}{2}}) \times X \times D(A^{\frac{1}{2}}) \times X$$

equipped with the following norm

$$\|(u, v, y, z)\|_{\mathcal{H}}^2 = a\left\|A^{\frac{1}{2}}u\right\|_X^2 + \left\|A^{\frac{1}{2}}y\right\|_X^2 + \|v\|_X^2 + \|z\|_X^2,$$

If we introduce $v = u_t$ and $z = y_t$, then we can write System (5) as an evolution equation in $\mathcal{H}$

$$\begin{cases} U_t(x,t) = \mathcal{B}U(x,t), \\ U|_{t=0} = U_0 := (u_0, u_1, y_0, y_1) \in \mathcal{H}, \end{cases} \tag{6}$$

where

$$D(\mathcal{B}) = \left\{ U = (u, v, y, z) \in \mathcal{H};\ v, z \in D(A^{\frac{1}{2}}),\ u, y \in D(A) \right\},$$

and

$$\mathcal{B}(u, v, y, z) = \left(v, -aAu - A^{\gamma}v - \alpha z, z, -Ay + \alpha v\right).$$

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$$E(t) = \frac{1}{2}\|(u, u_t, y, y_t)\|^2_{\mathcal{H}}$$

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On the other hand,

$$E(t) = \frac{1}{2}\|(u, u_t, y, y_t)\|_{\mathcal{H}}^2 \implies E'(t) = -\left\|A^{\frac{\gamma}{2}} u_t\right\|^2 \leq 0.$$

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We deduce that $\mathcal{B}$ generates a C_0-semigroup of contractions $e^{t\mathcal{B}}$ in $\mathcal{H}$ and therefore (6) is well-posed.

$a = 1$ and $\gamma = 0$, i.e.,

$$\begin{cases} u_{tt} + Au + u_t + \alpha y_t = 0, \\ y_{tt} + Ay - \alpha u_t = 0, \\ u|_{t=0} = u_0,\ u_t|_{t=0} = u_1,\ y|_{t=0} = y_0,\ y_t|_{t=0} = y_1. \end{cases} \tag{7}$$

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Since the resolvent of A is compact in X, there exists

1. $(\mu_n)_{n\geq 1}$ increasing sequence, $\mu_n \to \infty$;
2. orthonormal basis $(e_n)_{n\geq 1}$ of X

such that

$$Ae_n = \mu_n^2 e_n \quad \forall\, n \geq 1.$$

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Theorem "Ghader et al. published in CPAA journal 2018"

System (7) is exponentially stable, i.e., for all $U_0 \in \mathcal{H}$ there are constants $M \geq 1,\ \epsilon > 0$ such that

$$\left\| e^{t\mathcal{B}} U_0 \right\|_{\mathcal{H}} \leq M e^{-\epsilon t} \left\| U_0 \right\|_{\mathcal{H}}, \quad t \geq 0.$$

Sketch of the proof

Case 1. $\alpha^2 < \frac{1}{4}$;
$$\begin{cases} \lambda_{1,n}^{\pm} = \pm\, i\mu_n - \dfrac{1}{4} + \dfrac{1}{4}\sqrt{1-4\alpha^2} + o(1), \\ \lambda_{2,n}^{\pm} = \pm\, i\mu_n - \dfrac{1}{4} - \dfrac{1}{4}\sqrt{1-4\alpha^2} + o(1). \end{cases}$$

Case 2. $\alpha^2 = \frac{1}{4}$;
$$\begin{cases} \lambda_{1,n}^{\pm} = \lambda_{2,n}^{\pm} = \pm\, i\mu_n - \dfrac{1}{4} + o(1). \end{cases}$$

Case 3. $\alpha^2 > \frac{1}{4}$;
$$\begin{cases} \lambda_{1,n}^{\pm} = \pm\, i\mu_n + \dfrac{i}{4}\sqrt{4\alpha^2-1} - \dfrac{1}{4} + o(1), \\ \lambda_{2,n}^{\pm} = \pm\, i\mu_n - \dfrac{i}{4}\sqrt{4\alpha^2-1} - \dfrac{1}{4} + o(1). \end{cases}$$

Sketch of the proof

Therefore the eigenvalues $\left\{\lambda_{k,n}^{\pm}\right\}_{k=1,2,\ n\geq 1}$ of $\mathcal{B}$ are satisfy the following estimation:

$$\Re\left\{\lambda_{k,n}^{\pm}\right\} \sim -\frac{1}{4} \pm \frac{1}{4}\Re\left(\sqrt{1-4\alpha^2}\right) < 0.$$

On the other hand, we check that the system of eigenvectors of $\mathcal{B}$ forms a Riesz basis in $\mathcal{H}$.

Theorem "Ghader et al. published in CPAA journal 2018"

Assume that $a \neq 1$ or $\gamma \neq 0$, then for all $U_0 \in D(\mathcal{B})$, there exists $C > 0$ such that the energy of the system (5) has the **optimal polynomial decay rate**

$$E(t) \leq \frac{C}{t^{\delta(\gamma)}} \|U_0\|^2_{D(\mathcal{B})}, \quad \forall\, t > 0,$$

where

$$\delta(\gamma) = \begin{cases} -\frac{1}{\gamma}, & \text{if } a = 1 \text{ and } \gamma < 0, \\ \frac{1}{1-\gamma}, & \text{if } a \neq 1 \text{ and } \gamma \leq 0. \end{cases}$$

Sketch of the proof.

If $a = 1$ and $\gamma < 0$;

$$\begin{cases} \lambda_{1,n}^{\pm} \sim \pm\, i\mu_n + \dfrac{i\alpha}{2} + i(\cdots) - \dfrac{1}{4\mu_n^{-2\gamma}}, \\ \lambda_{2,n}^{\pm} \sim \pm\, i\mu_n - \dfrac{i\alpha}{2} + i(\cdots) - \dfrac{1}{4\mu_n^{-2\gamma}}. \end{cases}$$

then the real parts of $\lambda_{1,n}^{\pm}$ and $\lambda_{2,n}^{\pm}$ are of order $\mu_n^{2\gamma}$, hence the total energy decays at the optimal rate $t^{\frac{1}{\gamma}}$. □

Sketch of the proof.

If $a \neq 1$ and $\gamma \leq 0$;

$$\begin{cases} \lambda_{1,n}^{\pm} \sim \pm\sqrt{a} i\mu_n + +i(\cdots) - \dfrac{1}{2\mu_n^{-2\gamma}}, \\ \lambda_{2,n}^{\pm} \sim \pm\, i\mu_n + i(\cdots) - \dfrac{\alpha^2}{2(a-1)^2\mu_n^{2-2\gamma}}. \end{cases}$$

$$\Re\left\{\lambda_{1,n}^{\pm}\right\} \sim \mu_n^{2\gamma} \quad \text{and} \quad \Re\left\{\lambda_{2,n}^{\pm}\right\} \sim \mu_n^{2\gamma-2}$$

yielding a decay rate of the optimal total energy equal to $t^{-\frac{1}{1-\gamma}}$. □

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Example 1. Consider the System of weakly coupled wave equations

$$\begin{cases} u_{tt} - a\Delta u + (-\Delta)^{\gamma} u_t + \alpha y_t = 0 & \text{in } \Omega, \\ y_{tt} - \Delta y - \alpha u_t = 0 & \text{in } \Omega, \\ u = y = 0 & \text{on } \Gamma. \end{cases}$$

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We define the operator A in $L^2(\Omega)$ by

$$A = -\Delta \quad \text{with} \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Here, A is a **densely defined**, **closed**, and **self-adjoint** operator with compact resolvent in $L^2(\Omega)$.

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Hence, if $a = 1$ and $\gamma = 0$, we obtain an **exponential energy** decay rate given by:

$$\left\| e^{t\mathcal{B}} U_0 \right\|_{\mathcal{H}} \leq M e^{-\epsilon t} \left\| U_0 \right\|_{\mathcal{H}}, \quad t > 0.$$

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Hence, if $a = 1$ and $\gamma = 0$, we obtain an **exponential energy** decay rate given by:

$$\left\| e^{t\mathcal{B}} U_0 \right\|_{\mathcal{H}} \leq M e^{-\epsilon t} \left\| U_0 \right\|_{\mathcal{H}}, \quad t > 0.$$

Otherwise, we obtain an **optimal polynomial** energy decay rate of the form:

$$E(t) \leq \frac{C}{t^{\delta(\gamma)}} \left\| U_0 \right\|^2_{D(\mathcal{B})}, \quad \forall\, t > 0.$$

Example 2. Consider the System of weakly coupled plate equations

$$\begin{cases} u_{tt} + a\Delta^2 u + \left(\Delta^2\right)^{\gamma} u_t + \alpha y_t = 0 & \text{in } \Omega, \\ y_{tt} + \Delta^2 y - \alpha u_t = 0 & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = y = \frac{\partial y}{\partial n} = 0 & \text{on } \Gamma. \end{cases}$$

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We define the operator A in $L^2(\Omega)$ by

$$A = \Delta^2 \text{ with } D(A) = H^4(\Omega) \cap H_0^2(\Omega).$$

Here, A is a **densely defined**, **closed**, and **self-adjoint** operator with compact resolvent in $L^2(\Omega)$.

Example 2. Consider the System of weakly coupled plate equations

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Otherwise, we obtain

$$E(t) \leq \frac{C}{t^{\delta(\gamma)}} \|U_0\|^2_{D(\mathcal{B})}, \quad \forall\, t > 0.$$

$$\begin{cases} u_{tt} + Au + Bu_t + \alpha y = 0, \\ y_{tt} + Ay + \alpha u = 0, \end{cases} \qquad \begin{cases} u_{tt} + Au + Bu_t + \alpha y_t = 0, \\ y_{tt} + Ay - \alpha u_t = 0. \end{cases}$$

1. A is a self-adjoint, coercive operator with a compact resolvent in a separable Hilbert space X.
2. B is a self-adjoint operator B, such that $\rho_1 A^\gamma \leq B \leq \rho_2 A^\gamma$ ($0 < \rho_1 < \rho_2 < \infty$ and $\gamma < 0$).

$$\begin{cases} u_{tt} + A_1 u + A_1^{\gamma} u_t + \alpha y = 0, \\ y_{tt} + A_2 y + \alpha u = 0, \end{cases} \qquad \begin{cases} u_{tt} + A_1 u + A_1^{\gamma} u_t + \alpha y_t = 0, \\ y_{tt} + A_2 y - \alpha u_t = 0. \end{cases}$$

A_1 and A_2 are a self-adjoint, coercive operator with a compact resolvent in a separable Hilbert space X.

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The End.

Thank you for your attention! Questions?