Observabilty and stabilization of the wave equation with moving boundary

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We deal with the wave equation with assigned moving boundary ($0 < x < a(t)$) upon which Dirichlet or mixed boundary conditions are specified, here $a(t)$ is assumed to move slower than the light and periodically. We give a feedback which guarantees the exponential decay of the energy and prove an observation result.
The key to the results is the use of a reduction theorem of Yoccoz. [1]

1. J.C.Yoccoz,Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne, Ann.Sci.Éc.Norm.Supér.(4)17(1984),333–359.

We consider the following problems :

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < a(t),\ t > 0, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x),\ \ 0 < x < a(0), \end{cases} \tag{1}$$

$(\phi, \psi) \in H^1((0, a(0))) \times L^2((0, a(0)))$, with Dirichlet boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(a(t),t) = 0,\ t > 0, \tag{2}$$

or with mixed boundary conditions for which (2) is replaced by

$$u(0,t) = 0 \ \text{and} \quad u_x(a(t),t) = 0,\ t > 0, \tag{3}$$

the subscripts denote partial differentiations, here a is a strictly positive real function which is continuous, 1-periodic.

Our major concern will be to prove the following observability inequalities :

$$\int_0^T |u_x(a(t),t)|^2 \, dt \geq C^* \left(\|\phi\|^2_{H_0^1(0,a(0))} + \|\psi\|^2_{L^2(0,a(0))} \right), \quad (4)$$

for (1)-(2) and

$$\int_0^T |u_t(a(t),t)|^2 \, dt \geq C^* \left(\|\phi\|^2_{H_l^1(0,a(0))} + \|\psi\|^2_{L^2(0,a(0))} \right), \quad (5)$$

for (1)-(3), where
$H_l^1(0,a(0)) = \left\{ f \in H^1(0,a(0)) \text{ such that } f(0) = 0 \right\}$.

Note that if a is a constant, the observability inequality

$$\int_0^T |u_x(a,t)|^2 \, dt \geq C^* \left(\|\phi\|^2_{H_0^1(0,a)} + \|\psi\|^2_{L^2(0,a)} \right), \tag{6}$$

for some positive constant C^*, holds if $T \geq 2a$ for the Dirichlet problem. Also,

$$\int_0^T |u_t(a,t)|^2 \, dt \geq C^* \left(\|\phi\|^2_{H_l^1(0,a)} + \|\psi\|^2_{L^2(0,a)} \right), \tag{7}$$

holds for the mixed problem.

After we consider :

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < 1,\ t > 0, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x),\ \ 0 < x < 1, \end{cases} \tag{8}$$

$(\phi, \psi) \in H^1(0,1) \times L^2(0,1)$, with Dirichlet boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0,\ t > 0. \tag{9}$$

We give a complete description of all locations $a(t) \in (0,1)$ which guarantee the observability inequality (4).

Generalizations of the foregoing results may be obtained in a quasi periodic context and similar arguments can be repeated for that purpose.

Finaly, we consider the system :

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < a(t),\ t > 0, \\ u(0,t) = 0 \quad \text{and} \quad u_t(a(t),t) + f(t)u_x(a(t),t) = 0,\ t > 0, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x),\ \ 0 < x < a(0), \end{cases} \tag{10}$$

$(\phi, \psi) \in H_l^1((0, a(0))) \times L^2((0, a(0)))$ and where a is a strictly positive real function which is continuous, 1-periodic and $f \in L^\infty\left(\mathbb{R}_+^*\right)$.

Denote by

$$E_u(t) = \frac{1}{2}\int_0^{a(t)} \left[|u_t(x,t)|^2 + |u_x(x,t)|^2\right] dx$$

the energy of the field u. We describe the feedback $f(t)$ necessary to obtain the exponential decay of $E_u(t)$.

The mechanism of the problem was discovered by

J. Dittrich, P. Duclos, and N. Gonzalez, (1998) Stability and instability of the wave equation solutions in a pulsating domain, Rev. Math. Phys, 10(7), 925-962.

A detailed survey is given in

N. Gonzalez, (1997) Ph.D. Thesis, University of Toulon and Czech Technical University,

where they used a direct estimates on the energy.

Some notations and known results

We start with some notations and known results. Let Lip($\mathbb{R}$) be the space of Lipschitz continuous functions on $\mathbb{R}$. We shall denote the Lipschitz constant of a function F by

$$L(F) := \sup_{x,y\in\mathbb{R},x\neq y} \left| \frac{F(x) - F(y)}{x - y} \right| .$$

On the existence of solutions to the Dirichlet or the mixed problem, we refer the reader to :

N. Gonzalez, (1997) Ph.D. Thesis, University of Toulon and Czech Technical University.

We have the proposition :
If $a \in \text{Lip}(\mathbb{R})$, $L(a) \in [0, 1)$, $a > 0$ and

$$(\varphi_0, \varphi_1) \in H_0^1((0, a(0))) \times L^2((0, a(0))),$$

or in

$$H_l^1((0, a(0))) \times L^2((0, a(0))),$$

denote by $Q := (0, a(t)) \times \mathbb{R}_+$ and
$Q_\tau := (0, a(t)) \times (0, \tau), \tau \in \mathbb{R}_+$. There exists a unique weak solution [2] u

2. $u \in H^1(Q_\tau)$ is called a weak solution of either the Dirichlet or the mixed problem if $u_{tt} - u_{xx} = 0$ in $\mathcal{D}'(Q)$ and the boundary conditions are satisfied.

of either the Dirichlet or the mixed problem satisfying the initial conditions $u(x,0) = \phi(x), u_t(x,0) = \psi(x) \;\; 0 < x < a(0)$. Moreover there exists $f \in H^1_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that

$$u(x,t) = f(t+x) - f(t-x) \quad \text{a.e. in } Q, \tag{11}$$

and $u \in L^\infty(Q) \cap H^1(Q_\tau)$.

First, denote by D_p the set of continuous functions of the form $x + g(x)$, where $g(x)$ is a 1-periodic continuous function.

Proposition 1

Let a be a 1-periodic function. Then

$$F := (I + a) \circ (I - a)^{-1} \tag{12}$$

belongs to D_p. Moreover, the rotation number $\rho(F)$ defined by

$$\rho(F) = \lim_{n \to \infty} \frac{F^n(x) - x}{n}$$

exists, and the limit is equal for all $x \in \mathbb{R}$.

Herman, Michael R, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations
Publications Mathématiques de l'IHES, Tome 49 (1979), p. 5-233

Rigorous studies on accelerators pointing out the use of circle maps, were obtained by Pustyl'nikov.

L.D. Pustyl'nikov, Poincaré models, rigorous justification of the second element of thermodynamics on the basis of mechanics, and the Fermi acceleration mechanism, *Russian Math. Surveys* **50** (1995) 1 p. 145-189.

L.D. Pustyl'nikov, "A new mechanism for particle acceleration and a relativistic analogue of the Fermi-Ułam model", *Theoret. Math. Phys.* **77** (1988) 1 p. 1110-1115.

L.D. Pustyl'nikov, "A new mechanism of particle acceleration and rotation numbers", *Theoret. Math. Phys.* **82** (1990) 2 p. 180-187.

Of the papers also dealing with the subject, we indicate that of

N. P. Petrov, (2005) Methods of Dynamical Systems, Harmonic Analysis and Wavelets Applied to Several Physical Systems, Ph.D. Thesis, University of Texas at Austin.

N. P. Petrov, R. de la Llave and J. A. Vano, (2003) Torus maps and the problem of one-dimensional optical resonator with a quasiperiodically moving wall, Phys. D. 180, 140-184.

We also indicate that of

J. Cooper, (1993) Long-time behavior and energy growth for electromagnetic waves reflected by a moving boundary. IEEE Trans Antennas and Propagation, 41 (10), 1365-1370.

J. Cooper, (1993)Asymptotic behavior for the vibrating string with a moving boundary, *J. Math. Anal. Appl.* **174**, No.1, 67-87.

C. Gignoux, O. Meplan, (1995) Exponential growth of the energy of a wave in a 1D vibrating cavity. Application to the quantum vacuum (preprint ISN 95.24, Institut de Sciences Nucléaires, Grenoble).

The authors give sufficient conditions for the unbounded growth or the boundedness of the energy as $t \to \infty$.

After, we construct a transformation of the time-dependent domain $[0, a(t)] \times \mathbb{R}$ onto $[0, \rho(F)/2] \times \mathbb{R}$ that preserves the D'Alembertian form of the wave equation.

For this purpose, we use :

[Gonzalez] Assume that $a(t)$ is a 1-periodic function, $a(t) > 0$, $a \in \text{Lip}(\mathbb{R})$ such that $L(a) \in [0, 1)$. Assume also that $|a'(t)| < 1$ for all $t \in \mathbb{R}$ and $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$ such that there exists a function $H \in D_p$ and

$$H \circ F = H + \rho(F). \tag{13}$$

Before stating our main results, let us specify the following hypothesis on H.

Assumption 1

There exist $\lambda_1 > 0$ *and* $\lambda_2 > 0$ *such that*

$$\lambda_1 \leq H'(t) \leq \lambda_2, \; t \in \mathbb{R}. \tag{14}$$

The next example is giving in

N. Gonzalez, (1998) An example of pure stability for the wave equation with moving boundary, *J. Math. Anal. Appl.*, **228**, 51-59,
where Assumption 1 is guaranteed.
Let a be continuous and periodic on $\mathbb{R}$, $a > 0$, be such that

$$a(t) := \begin{cases} \alpha t + \frac{\alpha(1-\alpha)(1+\beta)}{2(\alpha-\beta)} & \text{if } \frac{\alpha(1+\beta)}{2(\alpha-\beta)} \leq t \leq \frac{\alpha(1+\beta)-2\beta}{2(\alpha-\beta)}, \\ \beta t - \beta + \frac{\alpha(1-\beta^2)}{2(\alpha-\beta)} & \text{if } \frac{\alpha(1+\beta)-2\beta}{2(\alpha-\beta)} \leq t \leq \frac{\alpha(3+\beta)-2\beta}{2(\alpha-\beta)}, \end{cases} \tag{15}$$

with $-1 < \beta < 0 < \alpha < 1$.

The function F is defined by :

$$F(x) := (I+a)\circ(I-a)^{-1}(x) = \begin{cases} l_1 x + F_0 & \text{if } 0 \le x \le x_0, \\ l_2 x + F_0 + 1 - l_2 & \text{if } x_0 < x < 1, \end{cases}$$

with $l_1 := \frac{1+\alpha}{1-\alpha}$, $l_2 := \frac{1+\beta}{1-\beta}$, $F_0 := \frac{l_2(l_1-1)}{l_1-l_2}$ and $x_0 := \frac{1-l_2}{l_1-l_2}$.

We extend F through the formula : $F(x+1) = F(x) + 1$ for any $x \in \mathbb{R}$.
Also the rotation number is given by the expression :

$$\rho(F) = \frac{\ln l_1}{\ln\left(\frac{l_1}{l_2}\right)}, \tag{16}$$

and the function H given by (13) is done by

$$H(x) = h_0 \ln(|x + h_1|) + h_2,$$

where $h_0 = \frac{1}{\ln\left(\frac{l_1}{l_2}\right)}$, $h_1 = \frac{l_2}{l_1 - l_2}$ and $h_2 = -\ln(h_1)$. H satisfies the inequalities :

$$\frac{1}{\ln(\frac{l_1}{l_2})} \frac{l_1 - l_2}{l_1} \leq H'(x) \leq \frac{1}{\ln(\frac{l_1}{l_2})} \frac{l_1 - l_2}{l_2}. \tag{17}$$

Theorem 1 (Neumann observability)

Under the Assumption 1 and for all $T \geq \rho(F)$ there exists a onstant $C^* > 0$ such that for all u solution of the system (1) with the Dirichlet boundary condition (2) and initial data $(\phi, \psi) \in H_0^1(0, a(0)) \times L^2(0, a(0))$, we have

$$\int_0^T |\, u_x(a(t), t)|^2 \, dt \geq C^* \left(\|\phi\|^2_{H_0^1(0,a(0))} + \|\psi\|^2_{L^2(0,a(0))} \right). \quad (18)$$

Remark 1

Here the observability time $\rho(F)$ is optimal, in the sense : if a is constant equal to a, it's given by $2a = \rho(F)$.

The exact controllability problem for the system

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < a(t),\ t > 0, \\ u(0,t) = 0 \quad \text{and} \quad u(a(t),t) = r(t),\ t > 0, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x) \ \ 0 < x < a(0) \end{cases} \tag{19}$$

at time T is the following : for each $(\phi, \psi) \in L^2(0, a(0)) \times H^{-1}(0, a(0))$, find $r \in L^2(0, T)$ such that the corresponding solution to (19) satisfies $u(., T) = 0, u_t(., T) = 0$ in $(0, a(T))$.

Based on the transformation mentioned above and for *a* given by (15), we get :

Corollary 1

There exists $r \in L^2(0,T)$ *such that the system (19) is exactly controllable at time* $T := |e^{\frac{\rho(F)-h_2}{h_0}} - h_1|$. *Note that T is optimal in the sense that this time is derived from an optimal time of observability.*

Proof

We begin, by defining a domain transformation $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, using H given by (13), as follows :

$$\begin{cases} \xi = (H(x+t) - H(-x+t))/2, \\ \tau = (H(x+t) + H(-x+t))/2, \end{cases} \tag{20}$$

for $(x, t) \in \mathbb{R}^2$.

Proposition 2

The transformation Φ *is a bijection of* $[0, a(t)] \times \mathbb{R}$ *to* $[0, \rho(F)/2] \times \mathbb{R}$ *and* Φ *maps the boundaries* $x = 0$ *and* $x = a(t)$ *onto the boundaries* $\xi = 0$ *and* $\xi = \rho(F)/2$.

Yamaguchi, M. ; Yoshida, H., Nonhomogeneous string problem with periodically moving boundaries, Fields Institute Communications, 25 (2000), 565-574.

Proposition 3

Let $u(x,t)$ *satisfying* $(\partial_t^2 - \partial_x^2)u(x,t) = 0$ *and* $V(\xi,\tau)$ *defined by* $u(\Phi^{-1}(\xi,\tau))$. *Then the following identity holds*

$$(\partial_t^2 - \partial_x^2)u(x,t) = K(\xi,\tau)(\partial_\tau^2 - \partial_\xi^2)V(\xi,\tau)$$

where $K(\xi,\tau)$ *is defined by*

$$4H' \circ H^{-1}(\xi+\tau)H' \circ H^{-1}(-\xi+\tau) \circ H^{-1}(\xi+\tau).$$

Henri Cabannes, Alain Haraux, Mouvements presque-périodiques d'une corde vibrante en présence d'un obstacle fixe, rectiligne ou ponctuel. International Journal of Non-Linear Mechanics Volume 16, Issues 5–6, (1981), Pages 449-458

The next lemma will be very useful for the proof of our main results.

Lemma 1

Denote by

$$E_u(t) = \frac{1}{2}\int_0^{a(t)} \left[|u_t(x,t)|^2 + |u_x(x,t)|^2\right] dx$$

the energy of the field u, and

$$E_V(\tau) = \int_0^{\rho(F)/2} \left(|V_\xi(\xi,\tau)|^2 + |V_\tau(\xi,\tau)|^2\right) d\xi,$$

the energy of the field V. Under the Assumption 1, there are two positive constants C_1 and C_2 such that

$$C_1 E_V(\tau) \le E_u(t) \le C_2 E_V(\tau). \tag{21}$$

Proof of Lemma 1.

We calculate,

$$\partial_t u = \partial_\xi V \partial_t \xi + \partial_\tau V \partial_t \tau \quad \text{and} \quad \partial_x u = \partial_\xi V \partial_x \xi + \partial_\tau V \partial_x \tau$$

and so,

$$\begin{aligned} E_u(t) &= \frac{1}{2} \int_0^{a(t)} \left[|u_t(x,t)|^2 + |u_x(x,t)|^2 \right] dx \\ &= \frac{1}{2} \int_0^{a(t)} \left\{ [V_\xi \xi_t + V_\tau \tau_t]^2 + |V_\xi \xi_x + V_\tau \tau_x|^2 \right\} dx. \end{aligned}$$

□

Make use of :

$$\xi_x = (\partial_x \xi) = (\partial_t \tau) = \tau_t = [H'(x+t) + H'(-x+t)]/2,$$

$$\xi_t = (\partial_t \xi) = (\partial_x \tau) = \tau_x = [H'(x+t) - H'(-x+t)]/2.$$

Hence,

$$\xi_t^2 + \xi_x^2 = \tau_t^2 + \tau_x^2 = \frac{1}{2}\left[\left|H'(x+t)\right|^2 + \left|H'(-x+t)\right|^2\right],$$

$$\xi_t \tau_t = \xi_x \tau_x = \frac{1}{4}[(H'(x+t))^2 - (H'(-x+t))^2].$$

Back to the energy,

$$
\begin{aligned}
E_u(t) &= \int_0^{a(t)} \frac{1}{4}\left[|V_\xi|^2 + |V_\tau|^2\right]\left[\left|H'(x+t)\right|^2 + \left|H'(-x+t)\right|^2\right] dx \\
&\quad + \int_0^{a(t)} \frac{1}{2}[V_\xi V_\tau]\left[\left|H'(x+t)\right|^2 - \left|H'(-x+t)\right|^2\right] dx \\
&= \int_0^{a(t)} \frac{1}{4}[|V_\xi|^2 + |V_\tau|^2 + 2V_\xi V_\tau]|H'(x+t)|^2 dx \\
&\quad + \int_0^{a(t)} \frac{1}{4}[|V_\xi|^2 + |V_\tau|^2 - 2V_\xi V_\tau]|H'(-x+t)|^2 dx.
\end{aligned}
$$

Also, differentiating $\xi = (H(x+t) - H(-x+t))/2$ on a slice $t =$constant, we obtain

$$d\xi = 1/2(H'(x+t) + H'(-x+t))dx.$$

Thanks to Assumption 1 and the inequality $|V_\xi V_\tau| \leq 1/2\left(V_\xi^2 + V_\tau^2\right)$ we get :

$$C_1 E_V(\tau) \leq E_u(t) \leq C_2 E_V(\tau), \tag{22}$$

for positive constants C_1, C_2.

Remark 2

Applying the transformation Φ, the system (1)-(2) becomes :

$$\begin{cases} \partial_\tau^2 V - \partial_\xi^2 V = 0, \quad \text{for} \quad 0 < \xi < \rho(F)/2, \, \tau \in \mathbb{R}, \\ V(0,\tau) = 0, \;\; V(\rho(F)/2,\tau) = 0, \;\; \tau \in \mathbb{R}, \\ V(\xi,0) = \phi_1(\xi), \;\; V_\tau(\xi,0) = \psi_1(\xi), \;\; \xi \in (0,\rho(F)/2). \end{cases} \tag{23}$$

We need the lemma.

Lemma 2

If $T \geq \rho(F)$*, there exists* $C(T) > 0$ *such that for all* $(\phi_1, \psi_1) \in H_0^1(0, \rho(F)/2) \times L^2(0, \rho(F)/2)$ *we have*

$$C(T) \int_0^T |V_\xi(\rho(F)/2, \tau)|^2 d\tau \geq \|\phi_1\|^2_{H_0^1(0,\rho(F)/2)} + \|\psi_1\|^2_{L^2(0,\rho(F)/2)}.$$

We consider (1)-(2) and state :

$$\partial_x u = \partial_\xi V \partial_x \xi + \partial_\tau V \partial_x \tau.$$

Next we have :

$$\partial_x u(a(t), t) = \partial_\xi V(\rho(F)/2, \tau)\partial_x \xi(a(t), t) + \partial_\tau V(\rho(F)/2, \tau).\partial_x \tau(a(t), t).$$

Since

$$\partial_x \xi = [H'(x+t)+H'(-x+t)]/2 \ \text{ and } \ \partial_x \tau = [H'(x+t)-H'(-x+t)]/2,$$

it follows that :

$$|\partial_x u(a(t),t)|^2 =$$

$$|\partial_\xi V(\rho(F)/2,\tau)\partial_x \xi(a(t),t) + \partial_\tau V(\rho(F)/2,\tau)\partial_x \tau(a(t),t)|^2$$

$$= \frac{1}{4}\left\{|\partial_\xi V(\rho(F)/2,\tau)[H'(x+t) + H'(-x+t)]\right\}^2.$$

Make use of Young inequalities, (18) is a consequence of Lemma 1 and Lemma 2.

Proof of Corollary 1.

Let us consider

$$\begin{cases} \partial_\tau^2 V - \partial_\xi^2 V = 0, \quad \text{for} \quad 0 < \xi < \rho(F)/2,\ \tau \in \mathbb{R}, \\ V(0,\tau) = 0, \ \ V(\rho(F)/2,\tau) = 0, \ \ \tau \in \mathbb{R}, \\ V(\xi,0) = V_0(\xi), \ \ V_\tau(\xi,0) = V_1(\xi), \ \ \xi \in (0,\rho(F)/2). \end{cases} \tag{24}$$

□

The system (24) is exactly observable at time $\rho(F)$, i.e., there exists $C > 0$ such that for all $T \geq \rho(F)$, we have

$$C(T) \int_0^T |V_\xi(\rho(F)/2, \tau)|^2 d\tau \geq \|\phi_1\|^2_{H_0^1(0,\rho(F)/2)} + \|\psi_1\|^2_{L^2(0,\rho(F)/2)},$$

and so the problem

$$\begin{cases} \partial_\tau^2 \tilde{V} - \partial_\xi^2 \tilde{V} = 0, & 0 < \xi < \rho(F)/2,\ \tau \in \mathbb{R}, \\ \tilde{V}(0,\tau) = 0, \quad \tilde{V}(\rho(F)/2,\tau) = g(\tau), \quad \tau \in \mathbb{R}, \\ \tilde{V}(\xi,0) = \tilde{V}_0(\xi), \quad \tilde{V}_\tau(\xi,0) = \tilde{V}_1(\xi), \quad \xi \in (0,\rho(F)/2) \end{cases} \tag{25}$$

is exactly controllable at $\rho(F)$, i.e., for all $(\tilde{V}_0, \tilde{V}_1) \in L^2(0,\rho(F)/2) \times H^{-1}(0,\rho(F)/2)$, there exists $g \in L^2(0,\rho(F))$ such that $\tilde{V}(\xi,\tau) = 0$ for all $\tau \geq \rho(F)$. Moreover, $g := V_\xi(\rho(F)/2,\tau)\chi_{(0,\rho(F))}(\tau)$.

So the transformed system

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, \qquad 0 < x < a(t),\ t \in \mathbb{R}, \\ u(0,t) = 0, \ \ u(a(t),t) = r(t), \ \ t \in \mathbb{R}, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \ \ x \in (0, a(0)) \end{cases}$$

is exactly controllable with a time of control $T := |e^{\frac{\rho(F)-h_2}{h_0}} - h_1|$ and a control $r(t)$ is given by $r(t) = g\left(\frac{H(a(t)+t)+H(-a(t)+t)}{2}\right)$.

In this section, we treat the Dirichlet observability. We assume :

Assumption 2

The function $b(t) := \dfrac{H'(a(t)+t) - H'(-a(t)+t)}{H'(a(t)+t) + H'(-a(t)+t)}$ *satisfies*

$$c_1 \leq b(t) \leq c_2, \quad c_1, c_2 > 0, \ \text{for all } t \in \mathbb{R}. \tag{26}$$

Note that for $a(t)$ giving by (15), we have :

$$b(t) = \frac{a(t)}{t + \frac{l_2}{l_1 - l_2}}.$$

This function satisfies for all $t \in \mathbb{R}$

$$2\, a_{min} \frac{\alpha-\beta}{1-\beta} \leq 2\, a(t) \frac{\alpha-\beta}{1-\beta} \leq b(t) \leq 2\, a(t) \frac{\alpha-\beta}{1+\beta} \leq 2\, a_{max} \frac{\alpha-\beta}{1+\beta}, \tag{27}$$

and Assumption 2 is satisfied.

Theorem 2 (Dirichlet observability)

Under the Assumptions 1 and 2 we have that for for all $T \geq \rho(F)$, there exists a constant $C^* > 0$ such that for all solution u of the system (1) with the mixed boundary condition (3) and initial data $(\phi, \psi) \in H_l^1(0, a(0)) \times L^2(0, a(0))$,

$$\int_0^T |u_t(a(t), t)|^2 \, dt \geq C^* \left(\|\phi\|^2_{H_l^1(0,a(0))} + \|\psi\|^2_{L^2(0,a(0))} \right). \quad (28)$$

Remark 3

The observability time is optimal here for the same reason as in Remark 1 and Theorem 1.

Remark 4

Using Φ given by (20), we transform the system (1)-(3) into :

$$\begin{cases} \partial_\tau^2 V - \partial_\xi^2 V = 0, \quad \textit{for} \quad 0 < \xi < \rho(F)/2, \ \tau \in \mathbb{R}, \\ V(0,\tau) = 0, \ \ V_\xi(\rho(F)/2,\tau) + b(t(\tau))V_\tau(\rho(F)/2,\tau) = 0, \\ V(\xi,0) = \phi_2(\xi), \ \ V_\tau(\xi,0) = \psi_2(\xi), \ \ \xi \in (0,\omega/2). \end{cases} \tag{29}$$

For the proof of Theorem 2, we need the following lemmas.

Lemma 3

There exist positive constants C and ω such that

$$E_V(\tau) \leq Ce^{-\omega\tau}E_V(0). \tag{30}$$

Proof of Lemma 3.

Define the Lyapunov function :

$$E_1(\tau) = \frac{1}{2}\int_0^{\rho(F)} [V_\xi^2(\xi,\tau) + V_\tau^2(\xi,\tau)]d\xi +$$

$$\delta \int_0^{\rho(F)} \xi V_\xi(\xi,\tau) V_\tau(\xi,\tau) d\xi.$$

We obtain for $\delta < \frac{1}{\rho(F)}$,

$$0 < (1 - \delta\rho(F))E_V(\tau) \leq E_1(\tau) \leq (1 + \delta\rho(F))E_V(\tau). \tag{31}$$

□

We derive E_1 with respect to τ, we get

$$E_1'(\tau) =$$

$$[V_\xi V_\tau]_{\xi=0}^{\xi=\rho(F)} - \frac{\delta}{2}\int_0^{\rho(F)} [V_\xi^2(\xi,\tau) + V_\tau^2(\xi,\tau)]d\xi + \frac{\delta}{2}[\xi(V_\xi^2 + V_\tau^2)]_{\xi=0}^{\xi=\rho(F)}$$

$$= [\frac{\delta}{2}(1 + b(t(\tau))^2) - b(t(\tau))] V_\tau^2(\rho(F),\tau) -$$

$$\frac{\delta}{2}\int_0^{\rho(F)} [V_\xi^2(\xi,\tau) + V_\tau^2(\xi,\tau)]d\xi.$$

We choose δ small enough, taking into account (27) and (31) we get

$$E_1'(\tau) \leq -\omega E_1(\tau).$$

The proof is complete.

Lemma 4

If $T \geq \rho(F)$*, then there exists* $C(T) > 0$ *such that for all* $(\phi_2, \psi_2) \in H_l^1(0, \rho(F)/2) \times L^2(0, \rho(F)/2)$ *we have*

$$C(T) \int_0^T |V_\xi(\rho(F)/2, \tau)|^2 \, d\tau \geq \|\phi_2\|^2_{H_l^1(0,\rho(F)/2)} + \|\psi_2\|^2_{L^2(0,\rho(F)/2)}, \tag{32}$$

and

$$C(T) \int_0^T |V_\tau(\rho(F)/2, \tau)|^2 \, d\tau \geq \|\phi_2\|^2_{H_l^1(0,\rho(F)/2)} + \|\psi_2\|^2_{L^2(0,\rho(F)/2)}. \tag{33}$$

Proof of Lemma 4.

The energy identity for the system (29) gives :

$$E_V(T) - E_V(0) = -\int_o^T b(t(\tau))|V_\tau(\rho(F)/2,\tau)|^2 d\tau.$$

Using (27) and (30), we obtain

$$\begin{aligned} \int_0^T |V_\tau(\rho(F)/2,\tau)|^2 \, d\tau &\geq C \int_0^T b(t(\tau))\, |V_\tau(\rho(F)/2,\tau)|^2 \, d\tau \\ &\geq C(E_V(0) - E_V(T)) \\ &\geq C\, E_V(0)\, (1 - e^{-\omega T}). \end{aligned}$$

This permit to conclude the second inequality in Lemma 4.
For the first inequality, it suffices to use (29) and (27).

□

For the proof of (28), we state as above :

$$\partial_t u = \partial_\xi V \partial_t \xi + \partial_\tau V \partial_t \tau.$$

Next we have :

$$\begin{aligned} |\partial_t u(a(t), t)|^2 &= \frac{1}{4}\{|\partial_\xi V(\rho(F)/2, \tau)[H'(x+t) - H'(-x+t)] \\ &+ \partial_\tau V(\rho(F)/2, \tau)[H'(x+t) + H'(-x+t)]\}^2. \end{aligned}$$

Make use of Young inequalities, Lemma 1, (32), (33) and (17), we obtain the desired result.

Observability of the wave equation with a moving interior point

Here, we consider the system :

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < 1,\ t > 0, \\ u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0,\ t > 0, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x),\ \ 0 < x < 1. \end{cases} \tag{34}$$

Let $a(t) \in (0,1)$. We prove the following observability estimate.

Theorem 3

Under the Assumption 1, there exist $T > 0$ and a constant $C^* > 0$ such that for all u solution of the system (34) with the Dirichlet boundary condition and initial data $(\phi,\psi) \in H_0^1(0,1) \times L^2(0,1)$, we have

$$\int_0^T |u_t(a(t),t)|^2 \, dt \geq C^* \left(\|\phi\|^2_{H_0^1(0,1)} + \|\psi\|^2_{L^2(0,1)} \right). \tag{35}$$

Remark 5

We mention that interior point observation in the case a is constant is not possible.

Applying the transformation Φ, the system (34) becomes :

$$\begin{cases} \partial_\tau^2 V - \partial_\xi^2 V = 0, \quad \text{for} \quad 0 < \xi < s(t\tau),\ \tau \in \mathbb{R}, \\ V(0,\tau) = 0, \ \ V(s(t),\tau) = 0, \ \ \tau \in R, \\ V(\xi,0) = \phi_1(\xi), \ \ V_\tau(\xi,0) = \psi_1(\xi), \ \ \xi \in (0, s(t)). \end{cases} \tag{36}$$

We denote $\alpha(t) := t + s(t)$ and $\beta(t) := t - s(t)$. These functions are both strictly increasing bijections from $\mathbb{R}_+$ to $[\pm s(0), \infty)$ respectively. We will also consider

$$\gamma = \alpha \circ \beta^{-1} : [-s(0), \infty) \to [s(0), \infty).$$

We develop the solution V of the system (36) into a series of the form

$$V(x,t) := \sum_{n\in\mathbb{Z}} A_n\left(e^{2i\pi n\varphi(t+x)} - e^{2i\pi n\varphi(t-x)}\right), \tag{37}$$

where the coefficients A_n are given by the initial data (ϕ_1, ψ_1). In order to satisfy the Dirichlet boundary condition, we need a solution φ to the functional equation

$$\varphi(t+s(t)) - \varphi(t-s(t)) = 1. \tag{38}$$

B. Haak and D. Hoang, Exact observability of a 1D wave equation on a non-cylindrical domain, SIAM J. Control. Optim. 57, (2019), 570–589.

Theorem 4

Assume that s is monotonic, φ of $\mathcal{C}^2$- class solution of (38). Suppose moreover that φ' is strictly decreasing when $s(.)$ is increasing or φ' is strictly increasing when $s(.)$ is decreasing. Then the solution V of (36) satisfy :

$$C_1 E_V(0) \leq \int_0^{\rho(F)/2+\gamma(-\rho(F)/2)} |V_\xi(\rho(F)/2,\tau)|^2 d\tau \leq C_2 E_V(0), \tag{39}$$

$$C_1 E_V(0) \leq \int_0^{\rho(F)/2+\gamma(-\rho(F)/2)} |V_\tau(\rho(F)/2,\tau)|^2 d\tau \leq C_2 E_V(0), \tag{40}$$

where C_1 et C_2 only depend on a.

In : Castro, Carlos, Exact controllability of the 1-d wave equation from a moving interior point
ESAIM : Control, Optimisation and Calculus of Variations, Tome 19 (2013) no. 1, p. 301-316,

the author gives several examples of curves $a(t)$ for exact controllability related to the system :

$$\begin{cases} u_{tt} - u_{xx} = f(t)\delta_{a(t)} \quad \text{for} \quad 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0,\ t > 0, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x), \ \ 0 < x < 1, \end{cases}$$

via the observability estimate :

$$\int_0^T \left| \frac{d}{dt}[\varphi(a(t),t)] \right|^2 dt \geq C^* \left(\|\phi\|^2_{H_0^1(0,1)} + \|\psi\|^2_{L^2(0,1)} \right), \quad (41)$$

where φ is the solution of the associated conservative system :

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0 \quad \text{for} \quad 0 < x < 1,\ t > 0, \\ \varphi(0,t) = \varphi(1,t) = 0,\ t > 0, \\ \varphi(x,0) = \phi(x), \varphi_t(x,0) = \psi(x),\ \ 0 < x < 1, \end{cases}$$

and T is given by an optical geometric condition requiring that any ray, starting anywhere in the domain and with any initial direction, must meet the dissipation zone before the time T.

Stabilization of the wave equation with moving boundary

In this section, we analyze the stabilization property of solutions for the wave equation with a moving boundary. Here, we consider the following system :

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < a(t),\ t > 0, \\ u(0,t) = 0 \quad \text{and} \quad u_t(a(t),t) + f(t)u_x(a(t),t) = 0,\ t > 0, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x),\ \ 0 < x < a(0), \end{cases} \tag{42}$$

$(\phi,\psi) \in H_l^1((0,a(0))) \times L^2((0,a(0)))$, where

$$H_l^1((0,a(0))) = \left\{ v \in H^1((0,a(0))),\ v(0) = 0 \right\}.$$

Here a is a strictly positive real function which is 1-periodic (i.e., $a(t+1) = a(t), \forall\, t > 0$) and $f \in L^\infty\left(\mathbb{R}_+^*\right)$.

Denote by

$$E_u(t) = \frac{1}{2}\int_0^{a(t)} \left[|u_t(x,t)|^2 + |u_x(x,t)|^2\right]\, dx$$

the energy of the field u. Our major concern will be to detect the feedback $f(t)$ necessary to obtain the exponential decay of $E_u(t)$.

On the existence of solutions to system (42), we refer the reader to Dittrich and al.

Here we prove the exponential stability for the solutions of (42).

Theorem 5 (Exponential stability)

Let

$$f(t) = \frac{(\mu - 1)H'(a(t) + t) + (\mu + 1)H'(-a(t) + t)}{(1 - \mu)H'(a(t) + t) + (\mu + 1)H'(-a(t) + t)} \tag{43}$$

where μ *is a nonnegative constant and assume that there exist* $\lambda_1 > 0$ *and* $\lambda_2 > 0$ *such that*

$$\lambda_1 \leq H'(t) \leq \lambda_2,\ t \in \mathbb{R}. \tag{44}$$

Then, in the case where $\mu \neq 1$*, there exists a positive constant* C *such that*

$$E_u(t) \leq C\, e^{-\omega t} E_u(0), \tag{45}$$

for every solution u *of (42) with initial data* $(\phi, \psi) \in H_l^1((0, a(0))) \times L^2((0, a(0)))$ *and where* $\omega = \ln(|\frac{1+\mu}{1-\mu}|)$.

In the case $\mu = 1$ which corresponds to $f(t) = 1$, we obtain

$$E_u(t) = 0, \text{ for all } t \geq T_0 =: \left(I + a\right)^{-1} \circ H^{-1}\left(\frac{3\rho(F)}{2}\right). \quad (46)$$

Proof of Theorem 5.

Now we consider the system :

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < a(t),\ t > 0, \\ u(0,t) = 0 \quad \text{and} \quad u_t(a(t),t) + f(t)\, u_x(a(t),t) = 0,\ t > 0, \\ u(x,0) = \phi_1(x), u_t(x,0) = \psi_1(x), \ \ 0 < x < a(0) \end{cases} \tag{47}$$

where $f(t) = \dfrac{(\mu-1)H'(a(t)+t) + (\mu+1)H'(-a(t)+t)}{(1-\mu)H'(a(t)+t) + (\mu+1)H'(-a(t)+t)}$ and μ is a nonnegative constant.

Proposition 4

The transformation of system (47) is

$$\begin{cases} v_{\tau\tau} - v_{\xi\xi} = 0 \quad \textit{for} \quad 0 < \xi < \rho(F)/2,\ \tau > 0, \\ v(0,\tau) = 0 \quad \textit{and} \quad v_\tau(\rho(F)/2,\tau) + \mu v_\xi(\rho(F)/2,\tau) = 0, \\ v(\xi,0) = \phi_1(\xi), v_\tau(\xi,0) = \psi_1(\xi), \ \ 0 < \xi < \rho(F)/2). \end{cases} \tag{48}$$

The stabilization of system (42) is a direct combination of Proposition 4, Lemma 1 and the following Lemma.

Lemma 5

For $\mu \neq 1$, there exists a positive constant C such that

$$E_V(\tau) \leq Ce^{-\ln\left(\left|\frac{1+\mu}{1-\mu}\right|\right)\tau} E_V(0),\ \forall\,\tau > 0, \tag{49}$$

where V is the solution of the following system :

$$\begin{cases} V_{\tau\tau} - V_{\xi\xi} = 0 \quad \text{for} \quad 0 < \tau < \rho(F)/2,\ \tau > 0, \\ V(0,\tau) = 0 \quad \text{and} \quad V_\tau(\rho(F)/2,\tau) + \mu V_\xi(\rho(F)/2,\tau) = 0, \\ V(\xi,0) = \phi_1(\xi),\ V_\tau(\xi,0) = \psi_1(\xi),\ \ 0 < \xi < \rho(F)/2. \end{cases} \tag{50}$$

It is well known that for $\mu = 1$, $E_V(\tau) = 0$ for all $\tau \geq \rho(F)$, see S. Cox and E. Zuazua, The Rate at which energy decays in a string damped at one end, Comm. Partial Dierential Equations., 19 (1994), 213–243.
for more details.

In a similar context, many authors of the Brasilian school investigate the existence and the exponential decay of solutions for variants of the wave equation in time-dependent increasing domains.

We consider the following problem :

$$\begin{cases} u_{tt} - u_{xx} + [f_1(t)u_t + f_2(t)u_x]\delta_{a(t)} = 0 \quad \text{for} \quad 0 < x < b(t), \\ u(0,t) = 0 \quad \text{and} \quad u(b(t),t) = 0,\ t > 0, \\ u(x,0) = \phi_1(x), u_t(x,0) = \psi_1(x) \ \ 0 < x < b(0), \end{cases} \tag{51}$$

where $\delta_{a(t)}$ denotes the Dirac mass concentrated in the point $a(t)$.

The aim is to determine the functions f_1, f_2 and b to get after transformation the vibrations of a string with the static pointwise damping and conclude the asymptotic behavior of the energy.

Proposition 5

The transformation of the system :

$$\begin{cases} 0 = u_{tt} - u_{xx} + K\left(\frac{H(a(t)+t)-H(-a(t)+t)}{2}, \frac{H(a(t)+t)+H(-a(t)+t)}{2}\right) \\ \cdot \left[\left(\frac{1}{H'(a(t)+t)} - \frac{1}{H'(-a(t)+t)}\right)u_t + \left(\frac{1}{H'(a(t)+t)} + \frac{1}{H'(-a(t)+t)}\right)u_x \right] \delta_{a(t)} \\ \quad \text{for} \quad 0 < x < b(t) = \Lambda_t^{-1}(1) - t, \; t > 0, \\ u(0,t) = 0 \quad \text{and} \quad u(b(t),t) = 0, \; t > 0, \\ u(x,0) = \phi_1(x), u_t(x,0) = \psi_1(x), \;\; 0 < x < b(0) \end{cases} \tag{52}$$

is

$$\begin{cases} v_{\tau\tau} - v_{\xi\xi} + v_\tau \delta_{\frac{\rho(F)}{2}} = 0 \quad \text{for} \quad 0 < \xi < 1, \ \tau > 0, \\ v(0,\tau) = 0 \quad \text{and} \quad v(1,\tau) = 0, \ \tau > 0, \\ v(\xi,0) = \phi(\xi), v_\tau(\xi,0) = \psi(\xi), \ \ 0 < \xi < 1, \end{cases} \tag{53}$$

where Λ_t is defined by $\Lambda_t(y) = \frac{H(y+t) - H(-y+t)}{2}$.

Remark 6

If we return to the Example 15 and after some computation we get that the transformation of the system :

$$\begin{cases} 0 = u_{tt} - u_{xx} + \left(\frac{8}{a(t)+t+h_1} \cdot \frac{a(t)+t}{-a(t)+t+h_1}\right) \cdot [a(t)u_t + (t+h_1)u_x]\, \delta_{a(t)} \\ \quad \textit{for} \quad 0 < x < b(t) = (t+h_1)e^{\frac{h_2}{h_0}} \tanh(\frac{1}{h_0}),\ t > 0, \\ u(0,t) = 0 \quad \textit{and} \quad u(b(t),t) = 0,\ t > 0, \\ u(x,0) = \phi_1(x), u_t(x,0) = \psi_1(x),\ \ 0 < x < b(0), \end{cases} \tag{54}$$

is system (53).

One can try to generalize the previous results to the case when a is no longer periodic but has some sort of quasiperiodicity :
A function $a(t), t \in \mathbb{R}$ is called quasiperiodic with basic frequencies $\omega = (\omega_1, ..., \omega_m) \in \mathbb{R}^m$ (briefly $2\pi/\omega-$q.p.) if there exists a continuous function $\hat{a}(\theta), \theta = (\theta_1, ..., \theta_m) \in \mathbb{R}^m$ that is 2π-periodic in each $\theta_i, i = 1, ..., m$ such that $a(t) = \hat{a}(\omega t)$ holds. $\hat{a}(\theta)$, is called the corresponding function and $\frac{2\pi}{\omega} = (\frac{2\pi}{\omega_1}, ..., \frac{2\pi}{\omega_m})$ is called the basic periods of a.

The problem is much more complicated, since there is no rotation number. However, in :

M. Yamaguchi, (2005) “One dimensional wave equations in domain with quasiperiodically moving boundaries and quasiperiodic dynamical systems”, J. Math. Kyoto Univ, Vol.45, No.1, 57-97,

the author uses a weaker notion of upper (resp. lower) rotation number of F at every point x as follows :

$$\overline{\rho}(F) = \limsup_{n \to +\infty} \frac{F^n(x) - x}{n}$$

$$\text{(resp. } \underline{\rho}(F) = \liminf_{n \to +\infty} \frac{F^n(x) - x}{n}\text{).}$$

As a consequence, it is shown that under the same Diophantine condition satisfied by $\overline{\rho}(F)$ (resp. $\underline{\rho}(F)$), the rotation number of F exists and coincides with the lower (resp. upper) rotation number.

Lemma 6

Assume that $a(t)$ *is an* $\eta-$*q.p function,* $\hat{a}(\theta)$ *is real analytic and satisfy* $|\hat{a}'(\theta)| < 1$ *for* $\eta, \theta \in \mathbb{R}^m$ *and set* $\beta = (\frac{2\pi}{\eta_1}, ..., \frac{2\pi}{\eta_m})$*. Assume also that there exists* $C_0 > 0$ *depending on* β *such that* $|(k, \beta) + \pi l/\overline{\rho}(F)| > \frac{C_0}{|k|^{m+1}}$*. Then, there exists a real analytic function* $H(\xi) = \xi + h(\xi)$*, where* $h(\xi)$ *is an* η*-q.p. function, such that*

$$H^{-1} \circ F \circ H(\xi) = \xi + \overline{\rho}(F). \tag{55}$$

Remark 7

Thanks to Lemma 6, all the privious results can be extended by similar arguments.

Many problems in sciences and engineering require new devices based on systems with moving boundaries. The corresponding mathematical models of these systems are initial-boundary-value-problems for the wave equation with moving boundary conditions at least at one moving boundary.