Lebanese University
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A New Proof Of Menger's Theorem

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hello,
مرحبا,

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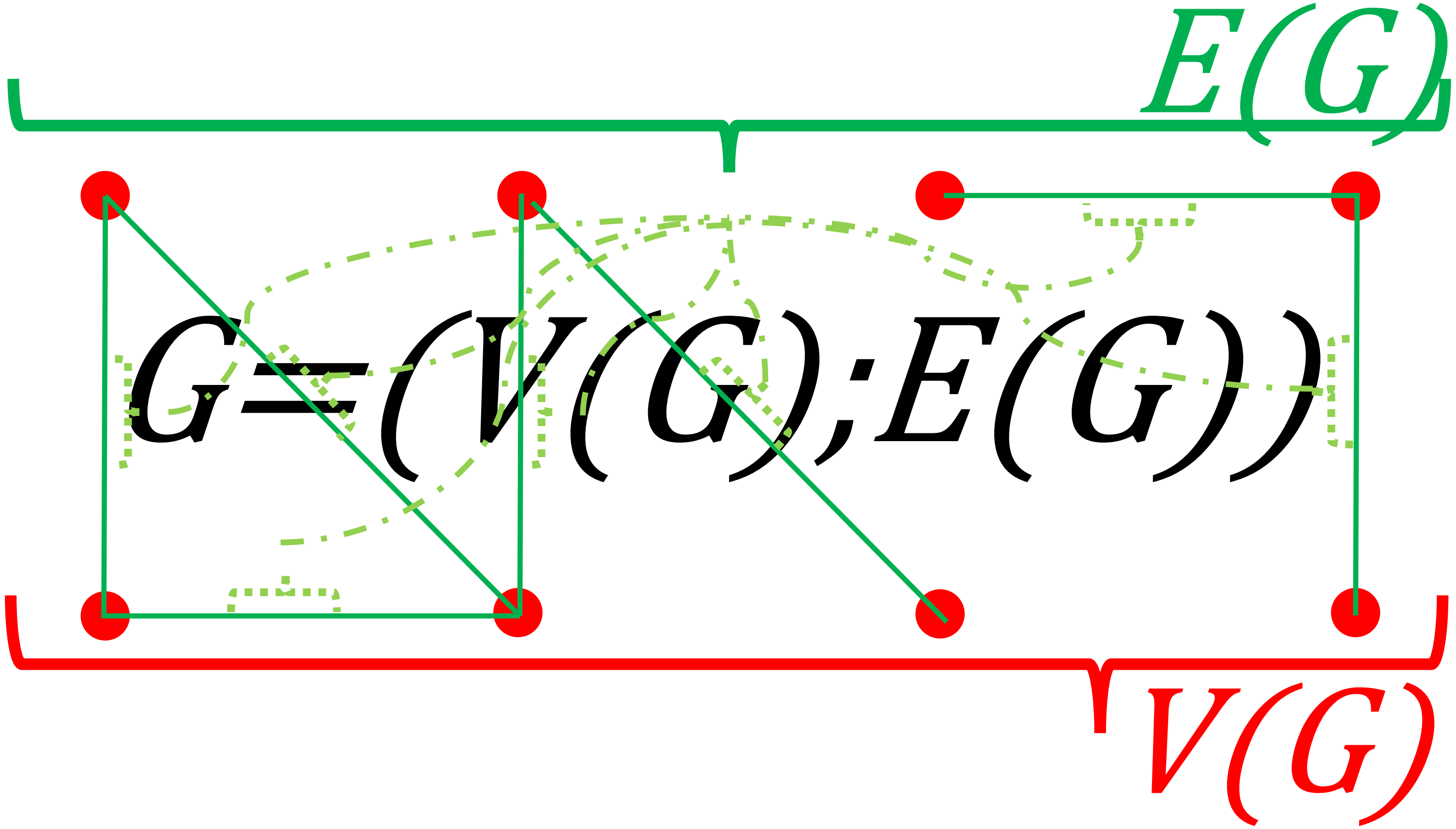
Outline:

- **Basic Defintions**
- **Minimal Separator**
- **Menger's Theorem**

Basic Definitions

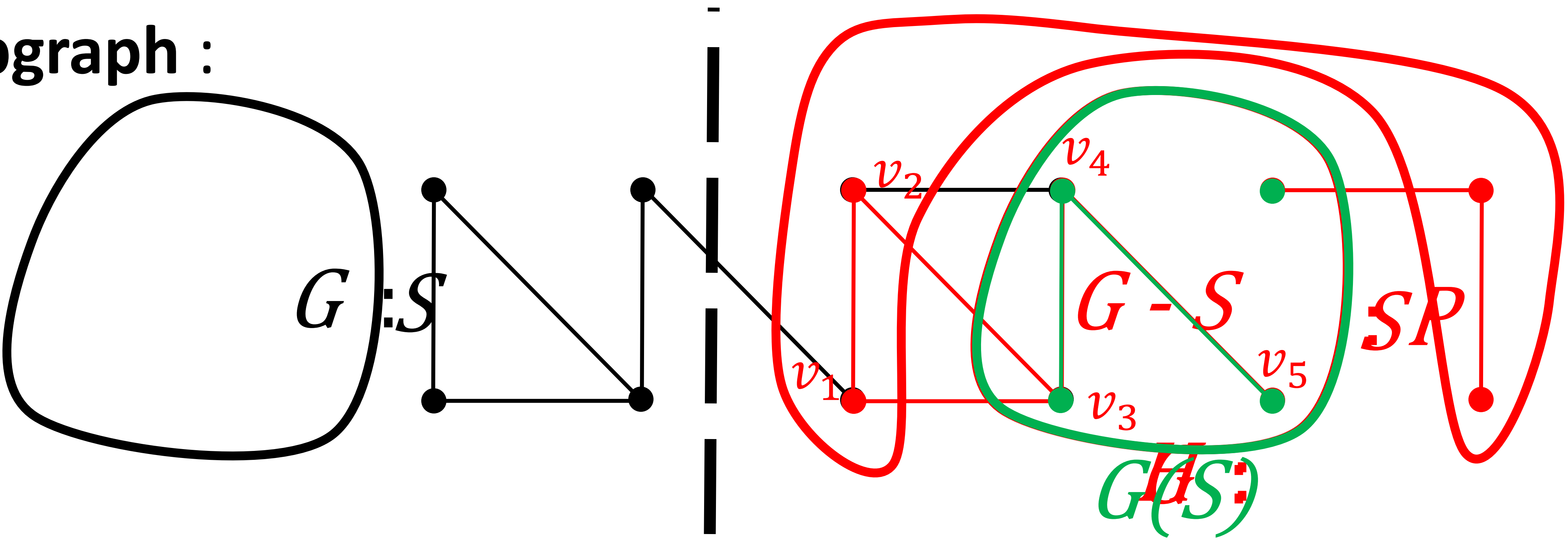
- **Graph :**

$$G = (V(G); E(G))$$



Basic Definitions

- **Subgraph :**

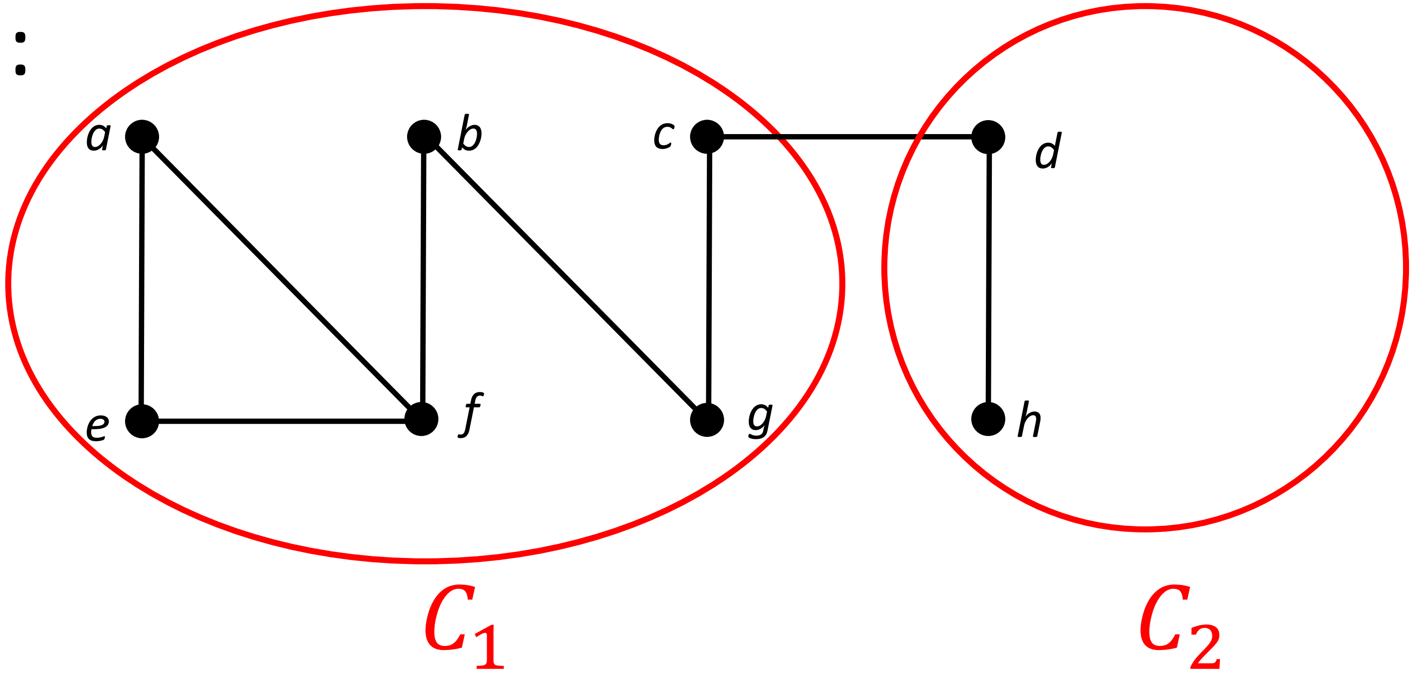


- A graph H is said to be a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
- A graph H is said to be an induced subgraph of G if $V(H) \subseteq V(G)$ and $\forall x, y \in V(H)$, $xy \in E(H) \Leftrightarrow xy \in E(G)$.
- Let S be a set of vertices of G. $V(P) = \{v_1, v_2, \ldots, v_s\}$ and $E(P) = \{v_1v_2, \ldots, v_{s-1}v_s\}$.
- Denoted by $G - S$ the subgraph of G induced by the set of vertices $V(G) - S$.
- Denoted by $G(S)$ the subgraph of G induced by S.

Basic Definitions

- **Connectivity :**

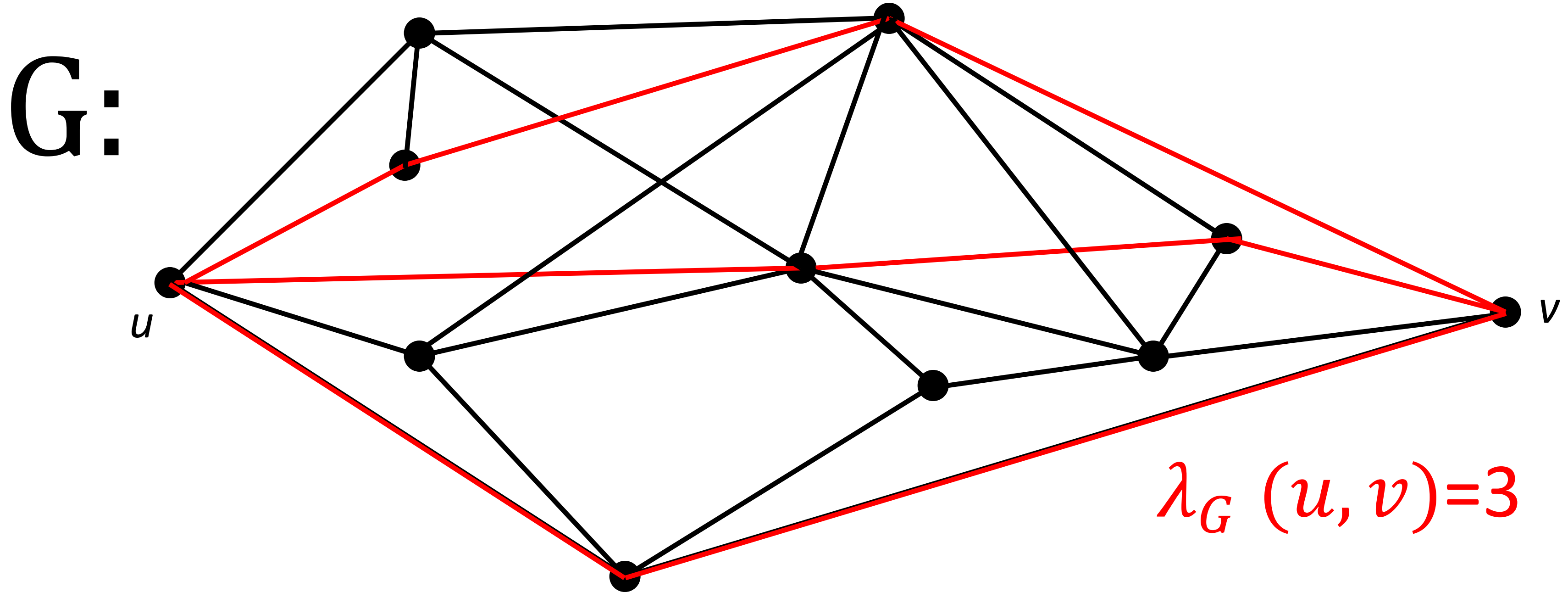
$G:$



- A graph G is said to be connected if $\forall u, v \in V\ (G)$, there exists an uv-path.
- Otherwise G is said to be disconnected, and a connected component of G is a maximal connected subgraph of G.

Basic Definitions

➢ **Disjoint Paths :**

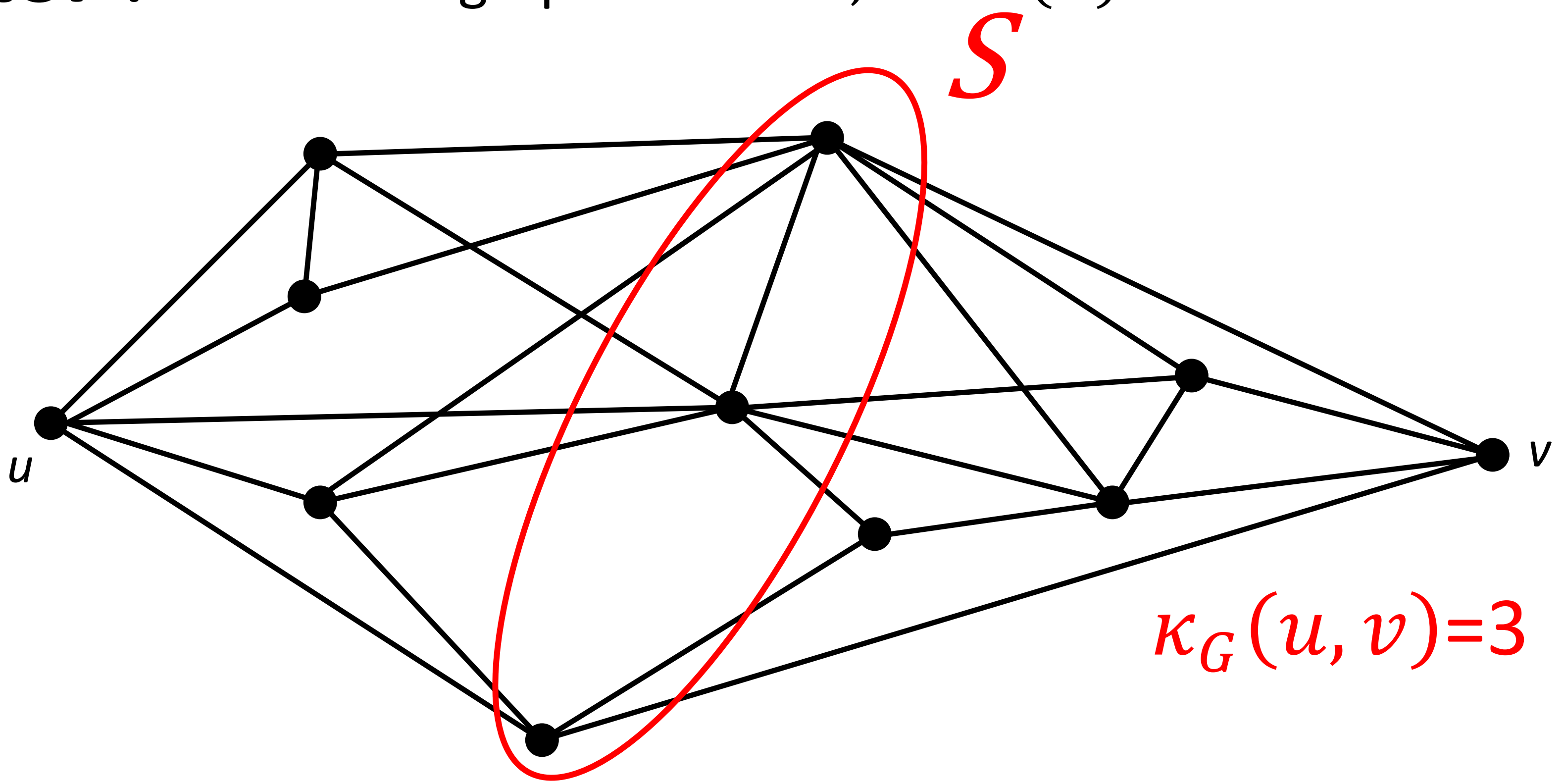


➢ Two uv-paths are internally disjoint if they have no vertices in common except u and v.

➢ The maximum number of internally disjoint uv-paths in a graph G is denoted by $\lambda_G(u, v)$.

Basic Definitions

➢ **Separator :** Let G be a graph and let $u, v \in V(G)$.



➢ A uv-separator of a graph G is a subset $S \subseteq V(G) - \{u, v\}$ such that $G - S$ has no uv-paths.

➢ The minimum size of a uv-separating set of G is denoted by $\kappa_G(u,v)$.

Outline:

- **Basic Defintions**
- **Minimal Separator**
- **Menger's Theorem**

Minimal Separator

- **Lemma 1: Consider a graph G, and let $u, v \in V(G)$ such that $uv \notin E(G)$. Then,**

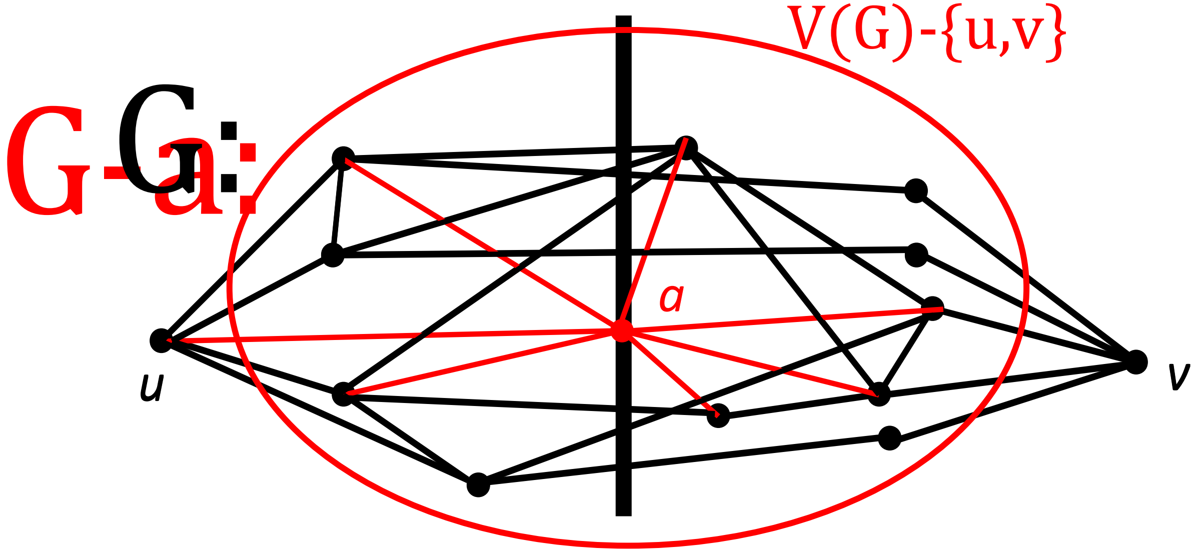
$$\kappa_G(u,v) - 1 \leq \kappa_{G-a}(u,v) \leq \kappa_G(u,v),$$

$$\forall a \in (V(G) - \{u,v\}) \cup E(G)$$

Minimal Separator

- **Case 1: If $a \in V(G) - \{u, v\}$**

- **Step 1:**

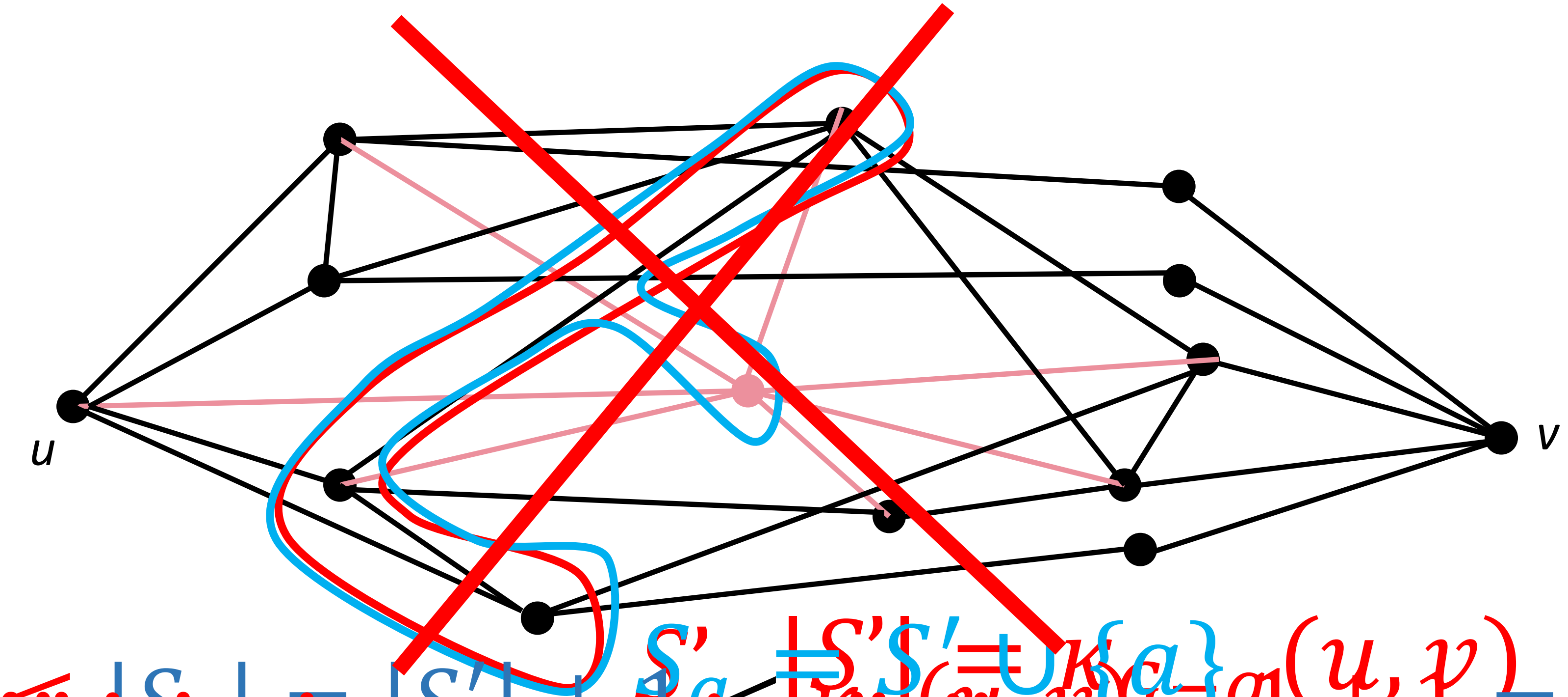


Clearly, then, $\kappa_{G-a}(u, v) \leq \kappa_G(u, v)$.

Minimal Separator

- **Case 1: If $a \in V(G) - \{u, v\}$**
  - **Step 2: Suppose that $\kappa_{G-a}(u,v) < \kappa_G(u,v) - 1$**

G:



$S'_a = S' \cup \{a\}$

So, $\kappa_G(u,v) \le |S_a| = |S'| + 1 = \kappa_{G-a}(u,v) + 1 < \kappa_G(u,v)$

So,

Which gives a contradiction

Minimal Separator

- Case 2: If $a = e \in E(G)$ is similarly to the case 1.

Then, $\kappa_G(u,v) - 1 \leq \kappa_{G-a}(u,v) \leq \kappa_G(u,v)$.

Then, $\kappa_G(u,v) - 1 \leq \kappa_{G-a}(u,v) \leq \kappa_G(u,v)$,

$\forall\ a \in (V(G) - \{u,v\}) \cup E(G)$.

The proof is complete.

Minimal Separator

- **Theorem 1: Consider a graph G, and let $u, v \in V(G)$ such that $uv \notin E(G)$. Then,**

$$\kappa_{G-e}(u,v) = \kappa_G(u,v), \quad \forall\, e \in G(S)$$

where S is any minimal uv-separator of G.

Outline:

- **Basic Defintions**
- **Minimal Separator**
- **Menger's Theorem**

Menger's Theorem

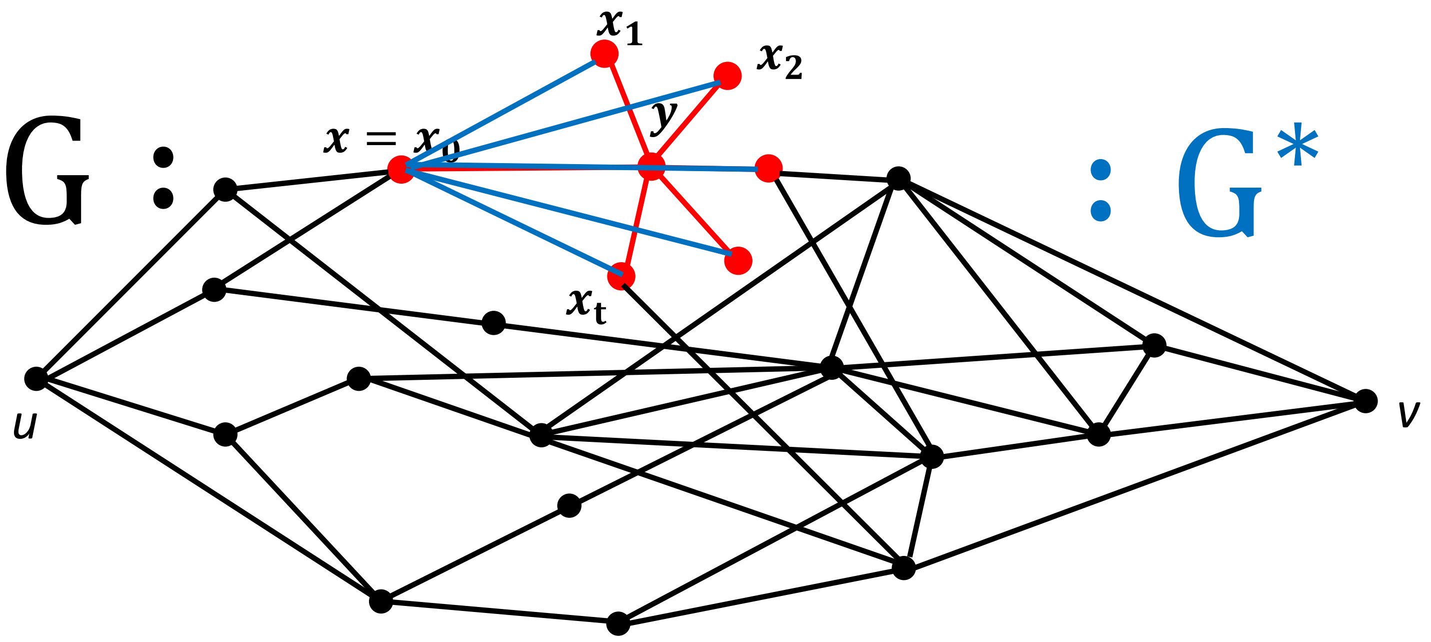
- **Theorem 1:**

Consider a graph G, and $u, v \in V(G)$ such that $uv \notin E(G)$. Then the size of a minimal uv-separator of G is equal to the maximum number of internally disjoint uv-paths in G; i.e. $\boldsymbol{\kappa_G(u, v) = \lambda_G(u, v)}$.

Menger's Theorem

- **Lemma 1:** Consider a graph G, and let $u, v, x, y \in V(G)$ such that $uv \notin E(G)$ and $xy \in E(G)$. Suppose that :

$$\kappa_{G-a}(u,v) = \kappa_G(u,v) - 1,\ \forall\, a \in (V(G) - \{u,v\}) \cup E(G).$$



Let $N(y) = \{x_0, x_1, \dots, x_t\}$ with $x_0 = x$.

Then, $\kappa_{G^*}(u, v) = \kappa_G(u, v)$.

Set: $G^* = G - y + \sum_{i=1}^{t} x_0 x_i$.

Menger's Theorem

- **Theorem 1: (Menger, 1927)**

Consider a graph G, and $u, v \in V(G)$ such that $uv \notin E(G)$. Then the size of a minimal uv-separator of G is equal to the maximum number of internally disjoint uv-paths in G; i.e. $\boldsymbol{\kappa_G(u,v) = \lambda_G(u,v)}$.

Menger's Theorem

Proof:

- Suppose that the statement is false, and let G be a graph with the least number of vertices such that $\kappa_G(u,v) = k$ and G contains no k internally disjoint uv-paths.
- G contains a spanning subgraph H which has $\kappa_H(u,v) = k$ but $\kappa_{H-e}(u,v) = k-1$ for all $e \in E(H)$; $G = H$ possibly.
- Clearly, we have:

$$\boldsymbol{\kappa_{G-a}(u,v) = \kappa_G(u,v) - 1,\ \forall\, a \in (V(G) - \{u,v\}) \cup E(G)}.$$

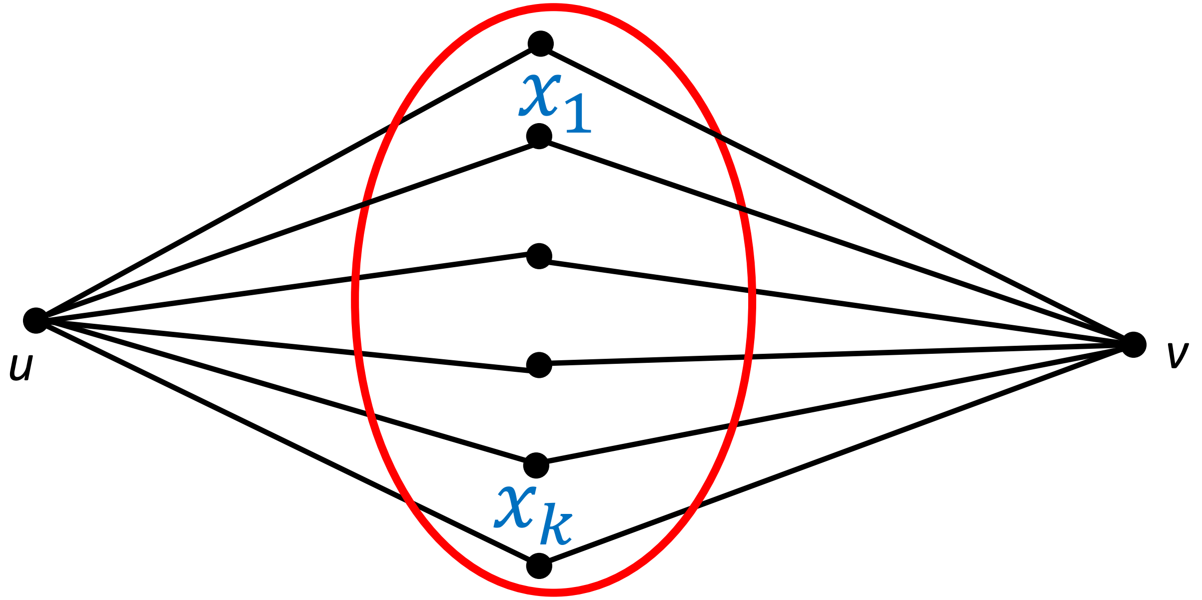
Menger's Theorem

Proof:

Claim : There exists x and $y \in V(G) - \{u, v\}$ such that $xy \in E(G)$.

Otherwise, we have:

G :

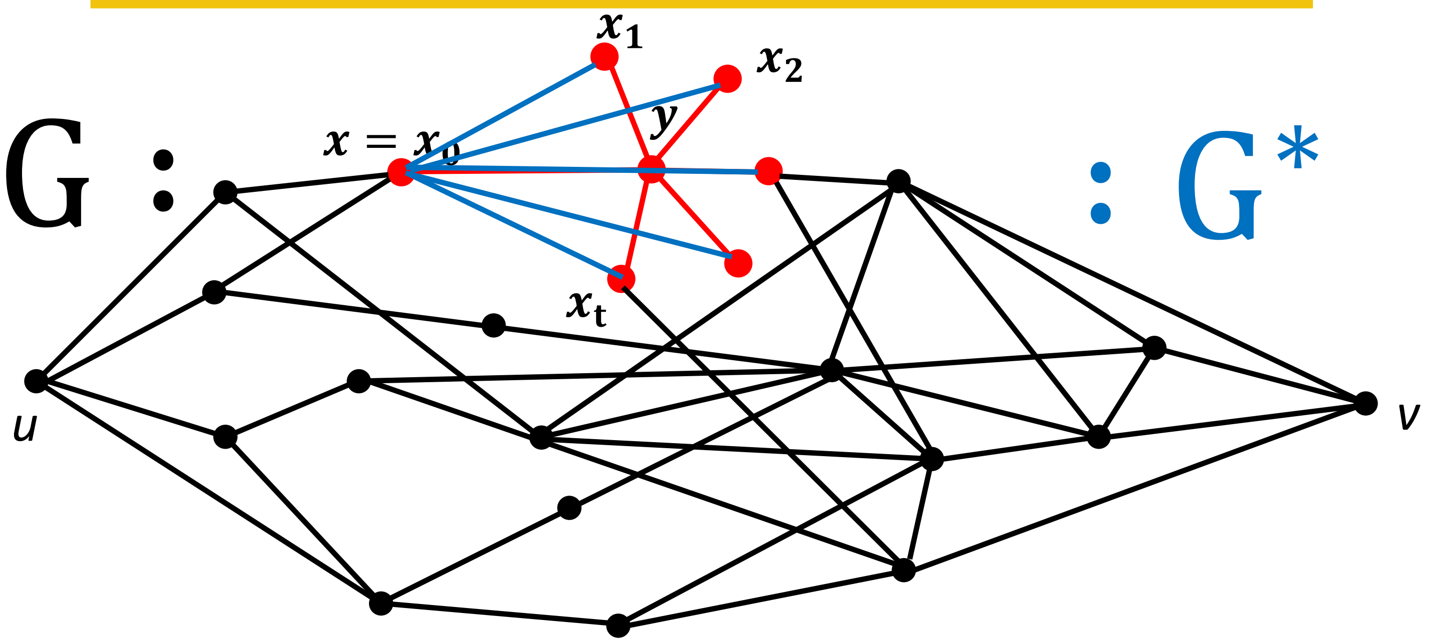


Because $\boldsymbol{\kappa_{G-a}(u, v) = \kappa_G(u, v) - 1, \forall a \in (V(G) - \{u, v\}) \cup E(G)}$.

Then, G contains k internally disjoint uv-paths; which gives a contradiction.

Menger's Theorem

Proof:



Let $N(y) = \{x_0, x_1, \ldots, x_t\}$ with $x_0 = x$.

By Lemma 1, we have $\kappa_{G^*}(u, v) = k$.

Set: $G^* = G - y + \sum_{i=1} x_0 x_i$.

Menger's Theorem

Proof:

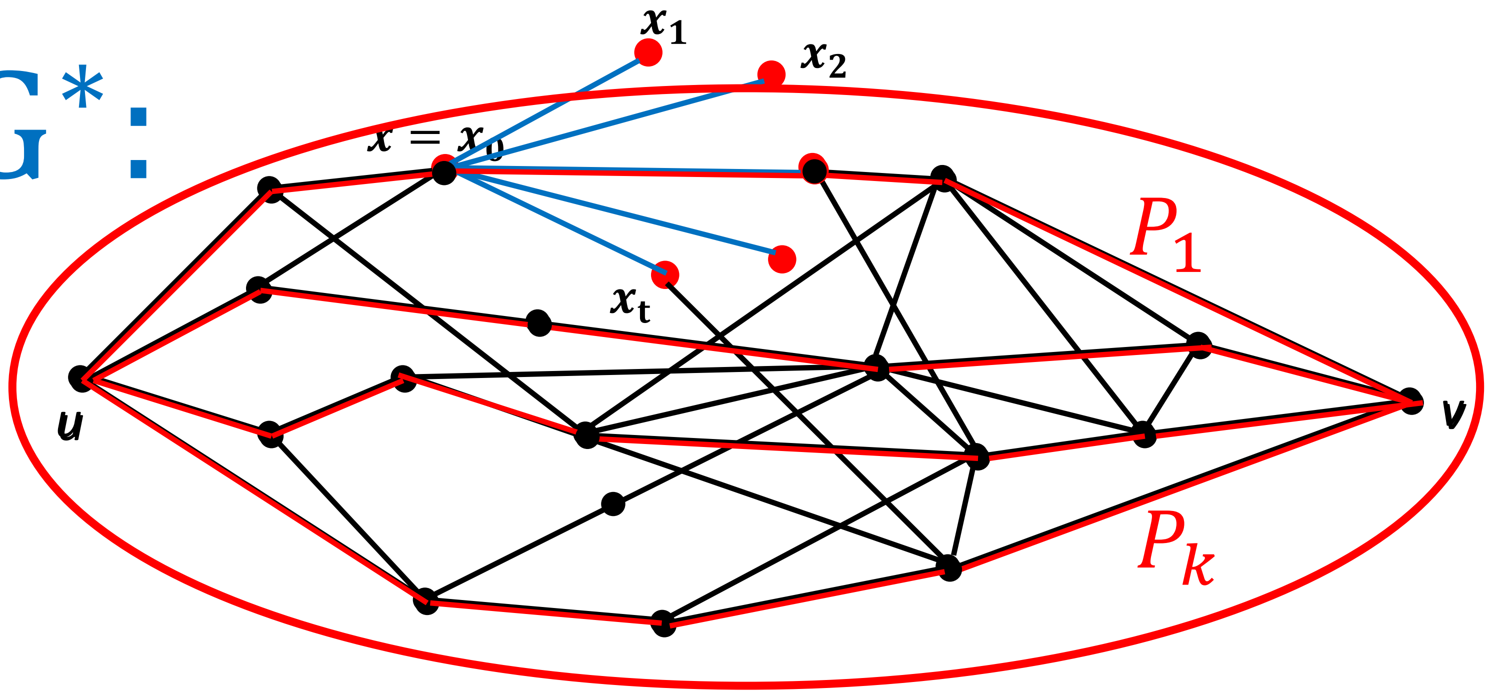
$\kappa_{G^*}(u, v) = k$

Then, by the minimality of $v(G)$ we have G^* contains k internally disjoint uv-paths.

Set

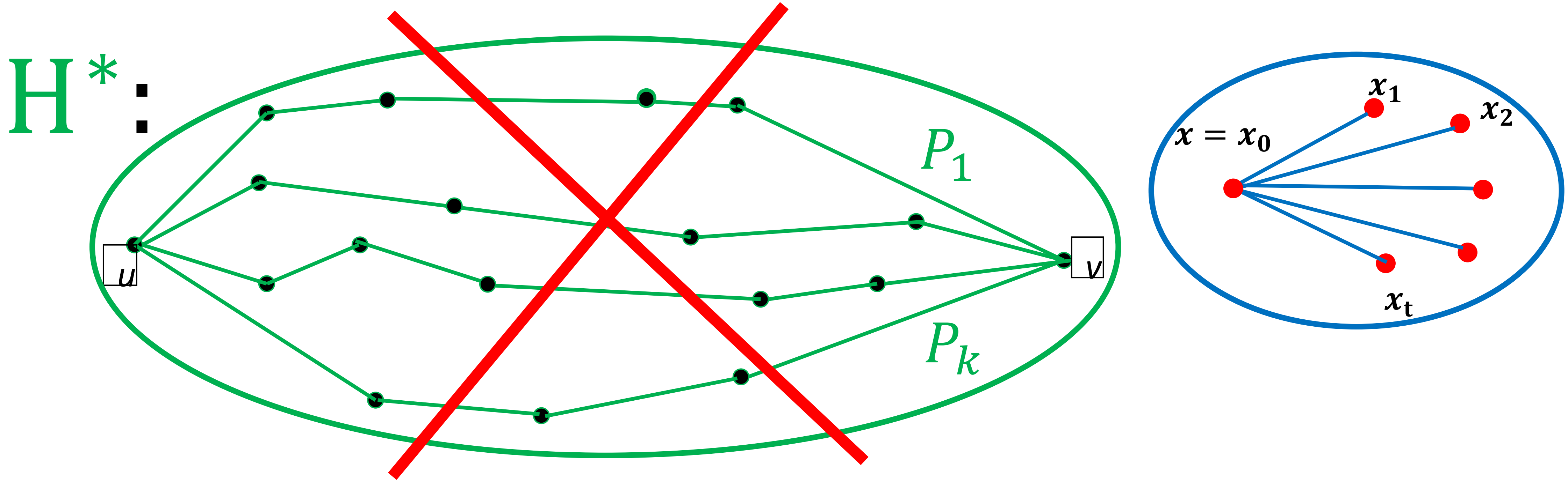
$H^* = \bigcup_{i=1}^{k} P_i$:

G^*:



Menger's Theorem

- If $E(H^*) \cap \{xx_i \mid 1 \leq i \leq t\} = \phi$



Which gives a contradiction

Menger's Theorem

Proof:

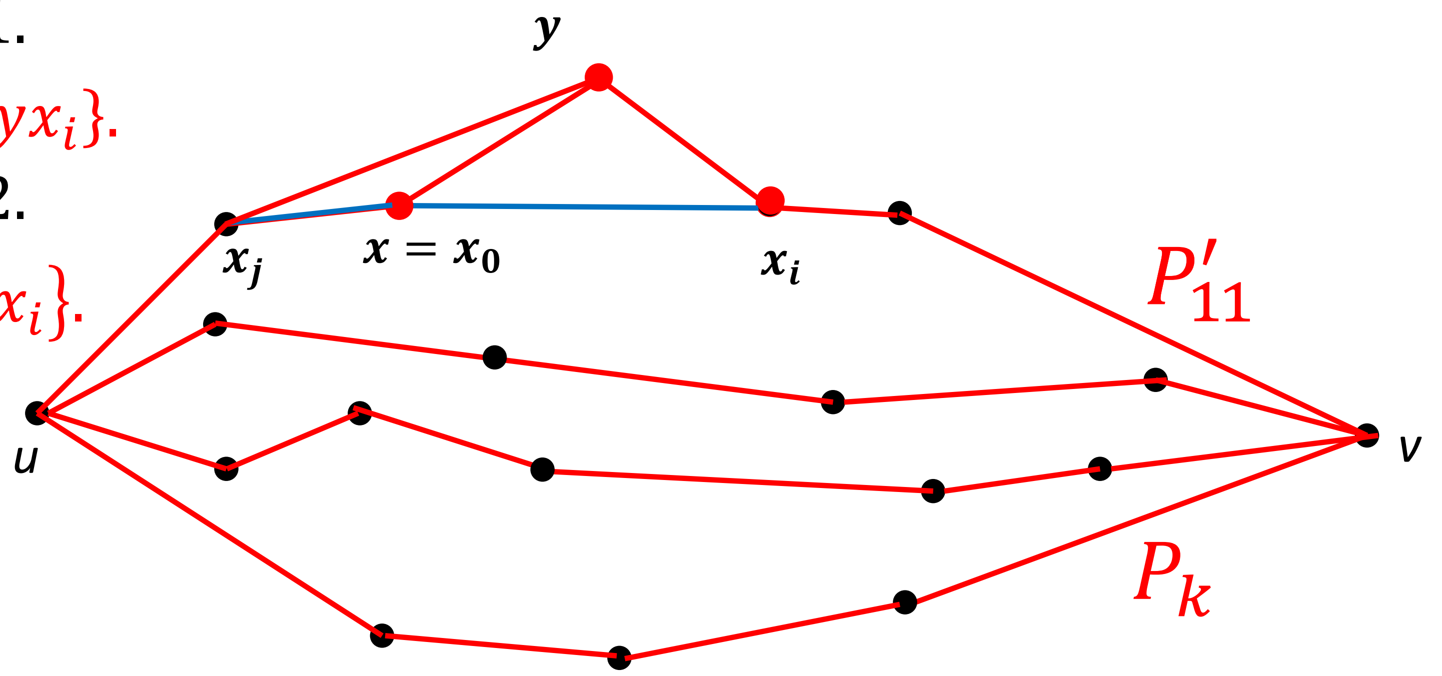
- Case 2: If H^* contains edges from $\{xx_i \mid 1 \le i \le t\}$. Then H^* contains at most two edges because $d_{H^*}(x) = 2$.

- If $|E(H^*) \cap \{xx_i \mid 1 \le i \le t\}| = 1$.

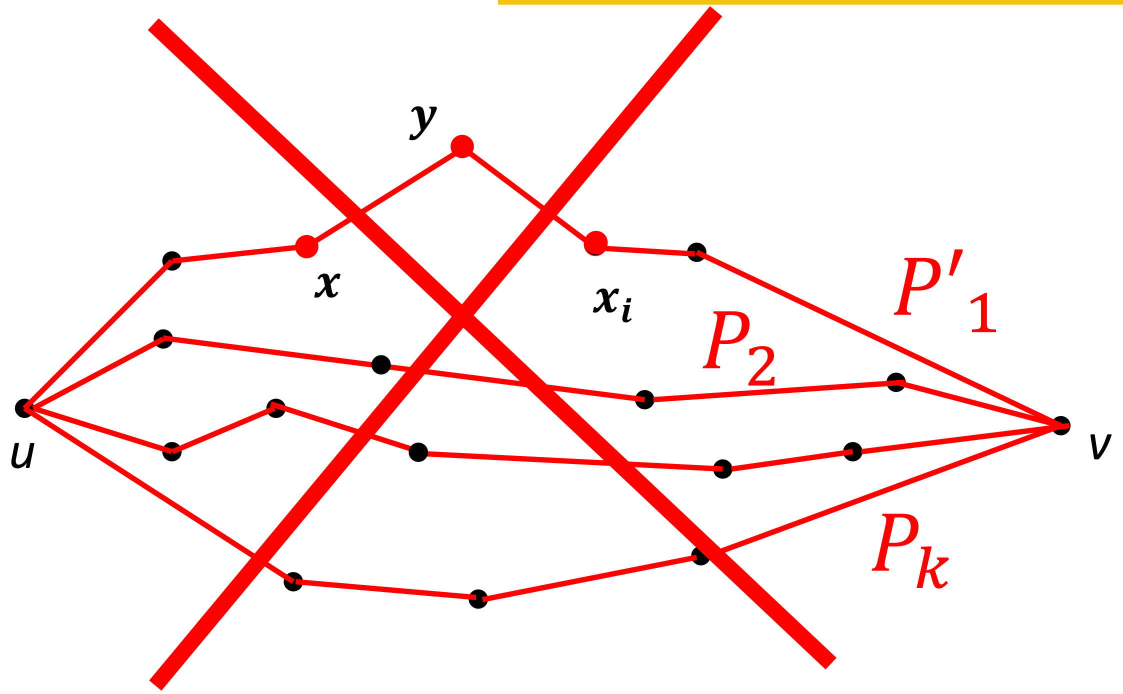
  Set $P_1' = (P_1 - xx_i) \cup \{y\} \cup \{xy, yx_i\}$.

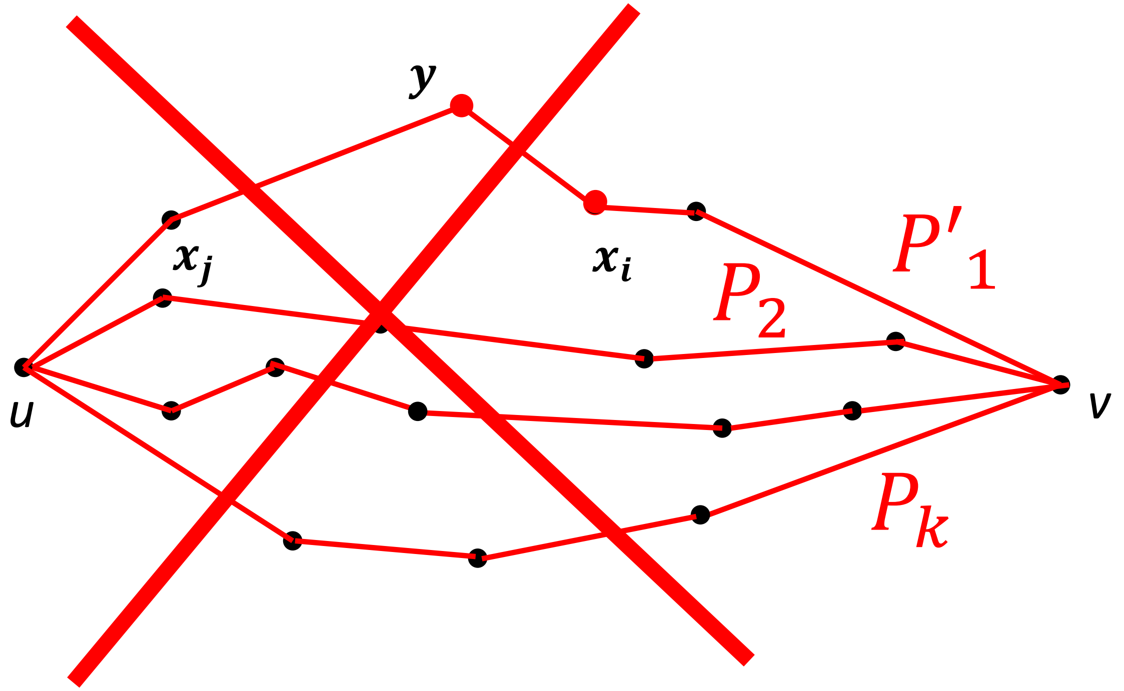
- If $|E(H^*) \cap \{xx_i \mid 1 \le i \le t\}| = 2$.

  Set $P_1' = (P_1 - x) \cup \{y\} \cup \{x_jy, yx_i\}$.



Menger's Theorem





In both cases, we have $\{P'_1, P_2, \ldots, P_k\}$ is a set of k internally disjoint uv-paths in G.

Which gives a contradiction

The proof is complete.

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