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Paneitz type equation

- Given (M, g) a smooth compact Einstein manifold of dimension $n \geq 5$ and let P_g be the Paneitz operator.

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- In this work, we show the attainability of the corresponding eigenvalues by generalized metrics.

- In 1983, Paneitz introduced a fourth-order conformally invariant operator on $n-$ dimensional Riemannian manifolds $n \geqslant 4$. Branson extended the notion of the $Q-$ curvature.

- More precisely, Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, and denote by Ric_g and S_g, the Ricci tensor and scalar curvature of g. For $u \in C^\infty(M)$, the Paneitz operator is given by

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- $$P_g u = \Delta_g^2 u - div_g(a_n S_g g + b_n Ric_g)^{\#} du + \frac{n-4}{2} Q_g u.$$

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- where $\Delta_g = -div_g(\nabla)$ is the Laplace-Beltrami operator,

$$a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)} \quad \text{and} \quad b_n = \frac{-4}{n-2}.$$

- The symbol $\#$ stands for the musical isomorphism (index are raised with the metric), and

$$Q_g = \frac{1}{n-2}\Delta_g S_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2}|Ric_g|^2.$$

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- In this section, we give some properties of the Paneitz operator.
- (1) The operator P_g is elliptic, self-adjoint with respect to the inner product in $L^2(M)$ and has a discrete spectrum :

$$\lambda_1(g) < \lambda_2(g) \leq \lambda_3(g) \leq \cdots \leq \lambda_k(g) \to +\infty$$

- (2) For any conformal metric, $\overline{g} = \varphi^{\frac{N-2}{2}} g$, $\varphi \in C^{\infty}(M)$, $\varphi > 0$ and $N = \frac{2n}{n-4}$ where the number N is the critical exponent of the Sobolev embedding $H_2^2(M) \subset L^N(M)$,

- (2) For any conformal metric, $\overline{g} = \varphi^{\frac{N-2}{2}} g$, $\varphi \in C^\infty(M)$, $\varphi > 0$ and $N = \frac{2n}{n-4}$ where the number N is the critical exponent of the Sobolev embedding $H_2^2(M) \subset L^N(M)$,
- the operator P_g is conformally invariant in the following sense:

for all $u \in C^{\infty}(M)$, we have

$$P_g(u\varphi) = \varphi^{N-1} P_{\overline{g}}(u),$$

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- (3) The quantity Q_g can be seen as the analogue of the scalar curvature for the conformal Laplacian and is called the Q-curvature.

- (4) A Riemannian manifold (M, g) is Einstein if and only if there exists a real number λ such that the Ricci tensor writes

$$Ric_g = \lambda g.$$

Here $\lambda = \frac{S_g}{n}$, where S_g is the scalar curvature and is constant in this case.
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$$P_g = \amalg_{l=1}^{2}(\Delta + c_l S_g) = \Delta^2 + (c_1 + c_2) S_g \Delta + c_1 c_2 S_g^2$$

where

$$c_l = \frac{(n + 2l - 2)(n - 2l)}{4n(n-1)}.$$

Moreover, if the scalar curvature S_g is positive :

Moreover, if the scalar curvature S_g is positive :

- (5) In the case of Einstein manifolds, the operator P_g is coercive.

(6) For all $u \in C^{\infty}(M)$ such that $P_g u > 0$, either $u > 0$ or $u \equiv 0$ and this statement is a direct consequence of comparison principle.
(7) It is not difficult to see that the quantity :

$$\int_M u P_g u dv_g = \int_M ([\Delta(u)]^2 + \sum_{l=0}^{1} a_l \left|\nabla^l u\right|^2) dv_g = \|u\|_{H_2^2}^2 ,$$

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$$\int_M u P_g u dv_g = \int_M ([\Delta(u)]^2 + \sum_{l=0}^{1} a_l \left|\nabla^l u\right|^2) dv_g = \|u\|^2_{H_2^2},$$

- where $a_l > 0$ is some constant, is a norm on $H_2^2(M)$ which is equivalent to the standard one :

$$\|u\|^2_{H_2^2} = \sum_{l=0}^{2} \int_M \left|\nabla^l u\right|^2 dv_g$$

Here $H_2^2(M)$ denotes the Sobolev space of functions u such that: u ,$|\nabla u|$ *and* $\left|\nabla^2 u\right| \in L^2(M)$. It is well known that by the Sobolev embedding theorem [Heb97] that $H_2^2(M) \subset L^q(M)$ where $1 < q \leq N = \frac{2n}{n-4}$ and this embedding is compact when $q < N$.

In this section, we quote some facts which will be used in the sequel of this paper: Grassmannians and the min-max principle. Let $L^N_+(M) = \{u \in L^N(M), \quad u \geq 0 \quad and \quad u \neq 0\}$

Definition

For all $u \in L_+^N(M)$, we define $Gr_p^u(H_k^2(M))$ as the set of all p-dimensional subspaces $(p \geq 1)$ of $H_2^2(M)$ that satisfy

$$span(v_1, ..., v_p) \in Gr_p^u(H_2^2(M))$$

if and only if $v_1, ..., v_p$ are linearly independent on $M \setminus u^{-1}(0)$.

Definition

A generalized metric conformal to g is a metric of the form $\overline{g} = u^{\frac{N-2}{k}} g$ with $u \in L^N_+$.

Definition

For any generalized metric $\overline{g} = u^{\frac{N-2}{2}} g$, the p^{th} eigenvalue $\lambda_p(\overline{g})$ of P_g is characterized by (see [1]):

$$\lambda_p(\overline{g}) = \inf_{V \in Gr_p^u(H_2^2(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M vP_g v dv_g}{\int_M u^{N-2} v^2 dv_g}, \qquad p \in \mathbb{N}^*.$$

Lemma

Let $u \in L_+^N(M)$ and let v_m be a sequence in $H_2^2(M)$ which converges weakly to u, then

$$\int_M u^{N-2}(\left|v_m^2 - v^2\right|)dv_g \to 0.$$

Let A be any large real number and set $u_A = inf(u, A)$. Then $(u_A)_A$ is a monotone sequence, which converges point-wise almost everywhere to u, so by the Lebesgue dominated convergence theorem, we have

$$\int_M (u^{N-2} - u_A^{N-2})^{\frac{N}{N-2}} dv_g \to 0 \text{ when } A \text{ tend to } +\infty.$$

On the other hand, we have

$$\int_M u^{N-2} \left| v_m^2 - v^2 \right| dv_g$$

$$\leq \int_M u_A^{N-2} \left| v_m^2 - v^2 \right| dv_g$$

$$+ (\int_M (u^{N-2} - u_A^{N-2})(|v_m| + |v|)^2 dv_g$$

By using Holder inequality, we can write :

$$\int_M u^{N-2} \left| v_m^2 - v^2 \right| dv_g$$

$$\leq A^{N-2} \int_M \left| v_m^2 - v^2 \right| dv_g$$

$$+ \left(\int_M (u^{N-2} - u_A^{N-2})^{\frac{N}{N-2}} dv_g \right)^{\frac{N-2}{N}} \left(\int_M (|v_m| + |v|)^N dv_g \right)^{\frac{2}{N}}$$

Since the sequence v_m is bounded in $H_2^2(M)$ then from Sobolev embedding, the boundedness in $L^N(M)$ is assumed, and hence there exists $C > 0$ such that $\int_M (|v_m| + |v|)^N dv_g \leq C$. By strong convergence of v_m in $L^2(M)$, we get the result.

Theorem

For any generalized metric $\overline{g} = u^{\frac{N-2}{2}} g$, *there exists a non trivial function* v *in* $H_2^2(M)$ *such that in the weak sense,* v *satisfies :*

$$P_g(v) = \lambda_{1,\overline{g}} u^{N-2} v \quad (1)$$

$$\text{and} \quad \int_M u^{N-2} v^2 dv_g = 1$$

where $\lambda_{1,\overline{g}}$ *is the first eigenvalue of* P_g *for the metric* $\overline{g}$. *In other words, the first eigenvalue of* P_g *is attained by* v.

Let (v_m) be a minimizing sequence for $\lambda_{1,\overline{g}}$, i.e a sequence $v_m \in H_2^2(M)$ such that $u^{\frac{N-2}{2}} v_m \neq 0$ and

$$\lim_m \frac{\int_M v_m P_g(v_m) dv_g}{(\int_M |u|^{N-2} v_m^2 dv_g)} = \lambda_{1,\tilde{g}}.$$

Without loss of generality, we can always normalize v_m by $\int_M u^{N-2} v_m^2 dv_g = 1$. Now for a large enough m, we have

$$\|u\|_{H_2^2}^2 = \int_M v_m P_g(v_m) dv_g \leq \lambda_{1,\tilde{g}} + 1,$$

then the sequence (v_m) is bounded in $H_2^2(M)$, and after restriction to a subsequence we may assume that there exists v in $H_2^2(M)$ such that $v_m \to v$ weakly in $H_2^2(M)$, strongly in $H_1^2(M)$ and in $L^2(M)$ and almost everywhere in M, so that

$$\int_M vP_g(v)dv_g \leq \liminf \int_M v_m P_g(v_m)dv_g = \lambda_{1,\tilde{g}}$$

and since $\lambda_{1,\tilde{g}}$ is the infimum, it follows that

$$\int_M vP_g(v)dv_g = \lambda_{1,\tilde{g}},$$

from lemma (4), we get

$$\int\limits_M u^{N-2}(v^2 - v_m^2)\, dv_g \to 0 \quad \text{i.e} \quad \int\limits_M u^{N-2}v^2 dv_g = 1.$$

Consequently v is a non-trivial weak minimizer of the functional associated to $\lambda_{1,\tilde{g}}$. Writing the Euler-Lagrange equation, we find that v satisfies the equation

$$P_g(v) = \lambda_{1,\tilde{g}} u^{N-2} v.$$

Moreover, we can also obtain the sign of the first eigenvalue in this case i.e :

$$\lambda_{1,\tilde{g}} = \|v\|^2_{H_2^2} > 0.$$

Proposition

If $u \in C_{+}^{\infty}(M)$, *then the solution of equation (1)* $v \in C^{2k}(M)$.

We have $\lambda_{1,\overline{g}} u^{N-2} v \in H_2^2(M)$, $P_g(v) \in H_2^2(M)$ and by regularity theorems $v \in H_{3k}^2(M)$, it follows by successive iterations that $v \in H_l^2(M)$ where l is large enough and finally if $\frac{1}{2} < \frac{l-m}{n}$,

$$H_l^2(M) \subset C^m(M)$$

so we can take $m = 4$ i.e

$$v \in C^4(M).$$

Now, we are going to show that the equation (1) has a positive solution.

Theorem

Let v be the solution of equation (1), there exists a non trivial positive function f in C^4, such that

$$P_g(f) = \lambda_{1,\overline{g}} u^{N-2} f \quad and \quad \int_M u^{N-2} f^2 dv_g = 1.$$

In other word, we can say that the first eigenvalue $\lambda_{1,\overline{g}}$ is attained by a $C^4(M)$ positive function.

Let v be a solution of (1) and let f be the solution of the equation

$$P_g(f) = |P_g(v)|,$$

we can show the existence of the function f by using the factorization of P_g as :

$$P_g = \Pi_{l=1}^{2}(\Delta + c_l S_g)$$

where c_l are positive, so all operators $\Delta + c_l S_g$ are invertible and applying strong maximum principle for elliptic equations of second order for 2 times(see [12] Proposition 4), we show that $f > |v| > 0$ and by regularity $f \in C^4(M)$.

Let A be a real number such that $0 < A \leq 1$ and $\int\limits_M (Af)^2 u^{N-2} dv_g = 1$, then

$$\int\limits_M (Af) P_g (Af) dv_g - \lambda_{1,\tilde{g}}$$

$$= A^2 \int\limits_M (f) \left| P_g(v) \right| dv_g - \lambda_{1,\tilde{g}}$$

$$= A^2 \left| \lambda_{1,\tilde{g}} \right| \int\limits_M (f) u^{N-2} \left| v \right| dv_g - \lambda_{1,\tilde{g}}$$

$$= A \left| \lambda_{1,\tilde{g}} \right| \int\limits_M (Af) u^{\frac{N-2}{2}} u^{\frac{N-2}{2}} \left| v \right| dv_g - \lambda_{1,\tilde{g}}$$

$$\leq A\,|\lambda_{1,\tilde{g}}|\left[\int_M (Af)^2 u^{N-2} dv_g\right]^{\frac{1}{2}}\left[\int_M (v)^2 u^{N-2} dv_g\right]^{\frac{1}{2}} - \lambda_{1,\tilde{g}}$$

$$\leq (A-1)\lambda_{1,\tilde{g}} \quad \text{as} \quad \lambda_{1,\tilde{g}} > 0.$$

Thus,

$$\int_M (Af)P_g(Af)dv_g \leq \lambda_{1,\tilde{g}}$$

and as $\lambda_{1,\tilde{g}}$ is the infimum, we get equality i.e Af is a positive solution of $P_g(v) = \lambda_{1,\tilde{g}} u^{N-2} v$.

Proposition

Let v be the solution of equation (1). Then the set :

$$E=\{w\in H_2^2(M) \quad such\ that$$

$$u^{\frac{N-2}{2}}w\neq 0\ ,\int\limits_M u^{N-2}w^2dv_g=1\ and\ \int\limits_M u^{N-2}wvdv_g=0\},$$

is not empty.

Let $v, s \in H_2^2(M)$ non-collinear functions, by multiplying if necessary v and s by certain constants, we assume that : $\int\limits_M u^{N-2} v^2 dv_g = \int\limits_M u^{N-2} s^2 dv_g = 1$, and thus $u^{\frac{N-2}{2}} v \neq 0$ and $u^{\frac{N-2}{2}} s \neq 0$. We set

$$w = \alpha v + \beta s$$

where α, β are real numbers.
Now we are going to find α, β such that $w \in E$. We begin by multiplying w by $u^{N-2} v$ and we integrate :

$$\int_M u^{N-2} wvdv_g = \alpha + \beta \int_M u^{N-2} svdv_g = 0$$

i.e

$$\beta = -\frac{\alpha}{\int_M u^{N-2} svdv_g}.$$

If $\int\limits_M u^{N-2} s v dv_g = 0$, then $s \in E$ and E is not empty and if $\int\limits_M u^{N-2} s v dv_g \neq 0$, hence β is well defined.
By the equality $\int\limits_M u^{N-2} w^2 dv_g = 1$, we obtain:
$\int\limits_M u^{N-2} (\alpha v + \beta s)^2 dv_g = 1 \quad$ i.e

$$\alpha^2 + \beta^2 + 2\alpha\beta \int\limits_M u^{N-2} s v dv_g = 1,$$

therefore

$$\alpha = \pm \frac{\int\limits_M u^{N-2} s v dv_g}{(1 - [\int\limits_M u^{N-2} s v dv_g]^2)^{\frac{1}{2}}}$$

then the number α is also well defined because $\int\limits_M u^{N-2} s v dv_g < 1$ is always true. In fact, if $\int\limits_M u^{N-2} s v dv_g \geq 1$, Holder inequality implies that

$$1 \leq \int\limits_M u^{N-2} s v dv_g = \int\limits_M u^{\frac{N-2}{2}} s u^{\frac{N-2}{2}} v dv_g$$

$$\leq [\int\limits_M u^{N-2} v^2 dv_g]^{\frac{1}{2}} [\int\limits_M u^{N-2} s^2 dv_g]^{\frac{1}{2}} \leq 1,$$

then there is equality in Holder inequality, and this is possible if and only if there is a real constant c such that $v = cs$, hence v and s are colinear and we get a contradiction.

Proposition

Let u and v two functions as in the theorem (5), then there exists a function w in $H_2^2(M)$ solution in the weak sense of the equation

$$P_g(w) = \lambda'_{2,g} u^{N-2} w,$$

such that $\int\limits_M u^{N-2} w^2 dv_g = 1$ *and* $\int\limits_M u^{N-2} wv dv_g = 0$ *where*

$$\lambda'_{2,g} = \inf_E \frac{\int_M vP_g v dv_g}{\int_M u^{N-2} v^2 dv_g}.$$

let $(w_m\ )$ be a minimizing sequence for $\lambda'_{2,g}$ i.e $w_m \in E$ is such that :

$$\lim_m \frac{\int_M w_m P_g(w_m) dv_g}{(\int_M u^{N-2} w_m^2 dv_g)} = \lambda'_{2,g},$$

with the same proof of theorem (5), we can find $w \in C^4(M)$ solution of $P_g(w) = \lambda'_{2,g}\ u^{N-2} w$ such that $\int\limits_M u^{N-2} w^2 = 1.$

Now writing

$$\int\limits_M u^{N-2} w v dv_g = \int\limits_M u^{N-2} w_m v - u^{N-2} w_m v + u^{N-2} w v dv_g$$

$$= \int_M u^{N-2} v(w - w_m) dv_g + \int_M u^{N-2} w_m v dv_g = 0.$$

As the sequence $w_m \in E$, $\int\limits_M u^{N-2} w_m v dv_g = 0$, and using the weak convergence of w_m to w in $L^N(M)$, we get

$$\int\limits_M u^{N-2} v(w - w_m) dv_g \to 0 \quad (u^{N-2} v \in L^{\frac{N}{N-1}} \text{ dual space of } \; L^N(M)).$$

Proposition

We have

$$\lambda'_{2,g} = \lambda_{2,\tilde{g}}$$

Since $w \in E$, the functions $u^{\frac{N-2}{2}}v$, $u^{\frac{N-2}{2}}w$ are linearly independent, then the space

$$V_0 = span(v, w) \in Gr_2^u(H_2^2(M)).$$

Putting $f = \alpha v + \beta w$ with α, β are non-zero real numbers, we evaluate

$$s = \frac{\int_M fP_g(f)dv_g}{\int_M u^{N-2}f^2 dv_g} \quad over \quad V_0$$

we find

$$s = \frac{\alpha^2 \int_M vL_g(v)dv_g + \beta^2 \int_M wP_g(w)dv_g}{\alpha^2 + \beta^2}$$

$$s = \frac{\alpha^2}{\alpha^2 + \beta^2}\lambda_{1,\tilde{g}} + \frac{\beta^2}{\alpha^2 + \beta^2}\lambda'_{2,\tilde{g}}$$

and as $\frac{\alpha^2}{\alpha^2+\beta^2} + \frac{\beta^2}{\alpha^2+\beta^2} = 1$, so for $\theta \in \mathbb{R}$, we can get :

$$s = \lambda_{1,\tilde{g}} \cos^2\theta + \lambda'_{2,\tilde{g}} \sin^2\theta.$$

On the other hand

$$\frac{ds}{d\theta} = (\lambda'_{2,\tilde{g}} - \lambda_{1,\tilde{g}}) \sin 2\theta,$$

and taking into account that:

$$\lambda_{1,\tilde{g}} = \inf_{H_2^2} \leq \lambda'_{2,\tilde{g}} = \inf_{E} \quad \text{because } (E \subset H_2^2(M)).$$

We get that $\lambda_{1,\tilde{g}}$ is a minimum of $s(\theta)$ and $\lambda'_{2,\tilde{g}}(\theta)$ is a maximum of s,

$$\lambda'_{2,\tilde{g}} = \sup_{f \in V_0} \frac{\int_M f P_g(f) dv_g}{\int_M u^{N-2} f^2 dv_g},$$

and as the infimum of the quantity $\sup\limits_{f\in V_0}\frac{\int_M fP_g(f)dv_g}{\int_M u^{N-2}f^2dv_g}$ on all elements of $Gr_2^u(H_2^2(M))$ is attained over V_0, it follows

$$\lambda'_{2,\tilde{g}} = \inf_{V\in Gr_2^u(H_2^2(M))} \sup_{v\in V\setminus\{0\}} \frac{\int\limits_M vP_g v dv_g}{\int\limits_M u^{N-2}v^2 dv_g} = \lambda_{2,\tilde{g}}.$$

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