Tuniso-Libanese workshop in Control Theory and Related Fields

Boundary Stabilization and Riesz Basis Generation of Two Strings Connected by a Point Mass with Variable Coefficients

Presented by: Walid BOUGHAMDA (FST)

October 30, 2019

Outline:

1. Introduction
2. Operator framework and well-posedness
3. Spectral analysis and stability results
4. Riesz basis generation

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\begin{aligned}
\rho_1(x)u_{tt}(x,t) = (\sigma_1(x)u_x(x,t))_x - q_1(x)u(x,t),\ \ x \in (-1,0), t > 0,\\
\rho_2(x)v_{tt}(x,t) = (\sigma_2(x)v_x(x,t))_x - q_2(x)v(x,t),\ \ x \in (0,1), t > 0,\\
Mz_{tt}(t) + \sigma_1(0)u_x(0,t) - \sigma_2(0)v_x(0,t) = 0,\ \ t > 0,\\
u(0,t) = v(0,t) = z(t),\ \ t > 0,
\end{aligned}
\tag{1}
$$

under the following initial conditions

$$\begin{aligned} u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), \quad & x \in (-1,0), \\ v(x,0) = v^0(x),\ v_t(x,0) = v^1(x), \quad & x \in (0,1), \\ z(0) = z^0,\ z_t(0) = z^1, & \end{aligned} \tag{2}$$

and the following boundary conditions

$$\begin{aligned} u(-1,t) = 0, \quad & t > 0, \\ \sigma_2(1) v_x(1,t) + K v_t(1,t) = 0, \quad & t > 0, \end{aligned} \tag{3}$$

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u(0,t) &= v(0,t) = z(t),\ t > 0,
\end{aligned}
$$

The functions $\rho_j(x)$ and $\sigma_j(x)$ represent respectively the density and the tension of each string, $j = 1, 2$. The potentials are denoted by the functions $q_1(x)$ and $q_2(x)$. The coefficients ρ_j and σ_j are assumed to be uniformly positive and q_j is nonnegative ($j = 1, 2$), and

$$\rho_1,\ \sigma_1 \in H^2(-1,0),\ q_1 \in L^1(-1,0),$$

$$\rho_2,\ \sigma_2 \in H^2(0,1),\ q_2 \in L^1(0,1).$$

$M > 0$ is the total mass concentrated in $x = 0$.

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and the following two types of boundary conditions

$$\begin{aligned} u(-1,t) = 0, &\quad t > 0,\\ \sigma_2(1) v_x(1,t) + K v_t(1,t) = 0, &\quad t > 0, \end{aligned}$$

$K > 0$, is the coefficient of damping..

- **S. Hansen and E. Zuazua**, (SIAM J. Control Optim, 1995)
- **W. Littman and S. W. Taylor**, (Nonlinear Problems in Mathematical Physics and Related Topic I, in: Int. Math., 2002).

- **S. Avdonin and J. Edward**, (SIAM J. Control Optim., 2018).
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The energy function of the above system is given by

$$E(t) = \frac{1}{2}\int_{-1}^{0} \left(\rho_1(x)|u_t|^2 + \sigma_1(x)|u_x|^2 + q_1(x)|u|^2\right) dx + \frac{M}{2}|z_t|^2 + \frac{1}{2}\int_{0}^{1} \left(\rho_2(x)|v_t|^2 + \sigma_2(x)|v_x|^2 + q_2(x)|v|^2\right) dx.$$

Formally

$$\frac{d}{dt}E(t) = -K\left|v_t(1,t)\right|^2 \leqslant 0.$$

We consider the Hilbert space $\mathcal{H}$ defined by

$$\mathcal{H} = \mathcal{W}_1 \times \mathcal{W}_0,$$

where, $\mathcal{W}_0 = L^2(-1,0) \times L^2(0,1) \times \mathbb{C}$, and

$$\mathcal{W}_1 = \left\{ \begin{array}{c} (u,v,z) \in H^1(-1,0) \times H^1(0,1) \times \mathbb{C}; \\ u(-1) = 0, u(0) = v(0) = z \end{array} \right\}.$$

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$$+\frac{1}{2}\int_{0}^{1} \left(\rho_2(x)|v_t|^2 + \sigma_2(x)|v_x|^2 + q_2(x)|v|^2\right) dx.$$

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By a standard reduction order method, System (1)-(3) can be rewritten as the first order evolution equation

$$\begin{cases} U_t(t) = \mathcal{A}U(t), \\ U(0) = U_0, \end{cases}$$

$U_0 = (u^0, v^0, z^0, u^1, v^1, z^1)^t$.

The operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ z \\ \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \\ \dfrac{1}{\rho_1(x)}((\sigma_1(x)u'(x))' - q_1(x)u(x)) \\ \dfrac{1}{\rho_2(x)}((\sigma_2(x)v'(x))' - q_2(x)v(x)) \\ \dfrac{1}{M}(-\sigma_1(0)u'(0) + \sigma_2(0)v'(0)) \end{pmatrix}, \qquad (4)$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{c} U = (u, v, z, \tilde{u}, \tilde{v}, \tilde{z})^t \in \mathcal{H}; \\ u \in H^2(-1,0), v \in H^2(0,1), z \in \mathbb{R}, \\ (\tilde{u}, \tilde{v}, \tilde{z}) \in \mathcal{W}_1, u(-1) = 0, \sigma_2(1)v_x(1) + K\tilde{v}(1) = 0 \end{array} \right\}. \qquad (5)$$

Theorem:

Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then, $\mathcal{A}$ generates a C_0-semigroup $T(t)$ on $\mathcal{H}$. Hence, the System (1)-(3) is well-posed, and the solution of (1)-(3) with initial data U_0 is given by $U(t) = T(t)U_0$.

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Lemma:

$\mathcal{A}^{-1}$ exists and is compact in $\mathcal{H}$. Hence, the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is discrete; i.e., it consists only of isolated eigenvalues with finite algebraic multiplicity: $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$, where $\sigma_p(\mathcal{A})$ denotes the set of eigenvalues of $\mathcal{A}$.

Eigenvalue problem:

$$\left\{\begin{array}{c} (\sigma_1(x)u'(x))' - q_1(x)u(x) = \lambda^2\rho_1(x)u(x), \qquad x \in (-1,0), \\ (\sigma_2(x)v'(x))' - q_2(x)v(x) = \lambda^2\rho_2(x)v(x), \qquad x \in (0,1), \\ u(-1) = 0, \quad \sigma_2(1)v'(1) = -K\lambda v(1), \\ u(0) = v(0), \\ -\sigma_1(0)u'(0) + \sigma_2(0)v'(0) = M\lambda^2 u(0). \end{array}\right. \tag{6}$$

Lemma:

$i\mathbb{R} = \{i\lambda_0; \lambda_0 \in \mathbb{R}\}$ is contained in $\rho(\mathcal{A})$.

Theorem:

The energy of System (1)-(3) decay asymptotically to zero.

In order to investigate the uniform stability of System (1)-(3), we need to know the asymptotic spectral properties of $\mathcal{A}$.

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In order to investigate the uniform stability of System (1)-(3), we need to know the asymptotic spectral properties of $\mathcal{A}$.

Proposition:

Assume that

$$K \neq \sqrt{\rho_2 \sigma_2(1)}. \tag{7}$$

Then there is a positive constant β such that

$$\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C} | \ -\beta \leq \Re e \lambda < 0\}.$$

Proposition:

Assume that (7) holds, then the set of eigenvalues $\{\lambda_n, n \in \mathbb{Z}\}$ of $\mathcal{A}$ satisfies the two asymptotic branches:

$$\begin{cases} \lambda_{n,1} = i\dfrac{n\pi}{\gamma_1} + \mathcal{O}(n^{-1}), \\ \lambda_{n,2} = \begin{cases} \dfrac{1}{2\gamma_2} \ln \left| \dfrac{\sqrt{\rho_2\sigma_2(1)} - K}{\sqrt{\rho_2\sigma_2(1)} + K} \right| + i \dfrac{\left(n + \dfrac{1}{2}\right)\pi}{\gamma_2} + \mathcal{O}(n^{-1}), \\ \qquad \text{if} \quad K < \sqrt{\rho_2\sigma_2(1)}, \\ \dfrac{1}{2\gamma_2} \ln \left| \dfrac{\sqrt{\rho_2\sigma_2(1)} - K}{\sqrt{\rho_2\sigma_2(1)} + K} \right| + i \dfrac{n\pi}{\gamma_2} + \mathcal{O}(n^{-1}), \\ \qquad \text{if} \quad K > \sqrt{\rho_2\sigma_2(1)}, \end{cases} \end{cases} \tag{8}$$

where $|n|$ is a sufficiently large integer, and

$$\gamma_1 = \int_{-1}^{0} \sqrt{\frac{\rho_1(s)}{\sigma_1(s)}} ds, \quad \text{and} \quad \gamma_2 = \int_{0}^{1} \sqrt{\frac{\rho_2(s)}{\sigma_2(s)}} ds. \tag{9}$$

Moreover, for large enough $|n|$, each eigenvalue λ_n is of algebraic multiplicity one.

Note that it is a necessary condition for exponential stability that the real part of spectrum point has a uniform negative upper bound. Hence, based on the distribution of the spectrum of $\mathcal{A}$, we know by (8) that there is a subsequence of eigenvalues of $\mathcal{A}$ which converges towards the imaginary axis. Then we have the following result on non-exponential stability of the system (1)-(3).

Corollary:

System (1)-(3) cannot achieve exponential stability.

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Definition:

A sequence $\{\phi_n : n \geqslant 1\}$ in a Hilbert space H is a Riesz basis if there is a linear bounded invertible operator T on H such that

$$T\phi_n = e_n, \quad n \geqslant 1,$$

where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of H.

Theorem:

Assume that (7) is fulfilled. Then the system of the (generalized) eigenfunctions of $\mathcal{A}$ is complete in $\mathcal{H}$.

Theorem: [G-Q. Xu and S. Yung, J. Diff. Equat., (2005)]

Let $\mathcal{H}$ be a separable Hilbert space, and $\mathcal{A}$ be the generator of a C_0-semigroup $T(t)$ on $\mathcal{H}$. Suppose that

(1) $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$, where $\sigma_2(\mathcal{A}) = \{\lambda_k\}_{k=1}^{\infty}$ consist of isolated eigenvalues of finite algebraic multiplicity.

(2) $\sup_{k\geq 1} m_a(\lambda_k) < \infty$, where $m_a(\lambda_k) = \dim E(\lambda_k, \mathcal{A})\mathcal{H}$ and $E(\lambda_k, \mathcal{A})$ is the Riesz projector associated with λ_k.

(3) There exists a constant α such that

$$\sup\{\Re e\lambda |\ \lambda \in \sigma_1(\mathcal{A})\} \leq \alpha \leq \inf\{\Re e\lambda |\ \lambda \in \sigma_2(\mathcal{A})\},$$

and

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0.$$

Then the following assertions are true:

(i) There exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with property that $\sigma_1(\mathcal{A}|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$ and $\sigma_2(\mathcal{A}|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$, and $\{E(\lambda_k, \mathcal{A})\mathcal{H}_2\}_{k=1}^{\infty}$ forms a Riesz basis of subspaces for $\mathcal{H}_2$. Furthermore, $\mathcal{H} = \overline{\mathcal{H}_1 \oplus \mathcal{H}_2}$

(ii) If $\sup_{k\geq 1} \|E(\lambda_k, \mathcal{A})\| < \infty$, then $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}$.

(iii) $\mathcal{H}$ has a decomposition of the topological direct sum, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, if and only if

$$\sup_{n\geq 1} \left\| \sum_{k=1}^{n} E(\lambda_k, \mathcal{A}) \right\|_{\mathcal{H}} < \infty.$$

Theorem:

Assume that Condition (7) holds. Then there is a sequence of the (generalized) eigenfunctions of $\mathcal{A}$ which forms a Riesz basis for $\mathcal{H}$.

Thank you for your attention