Stabilization of viscoelastic wave equation with dynamic boundary conditions

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Analysis and Control PDE

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Outline

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Our main interest lies in the following system of viscoelastic equation :

(1)

$$\begin{cases} u_{tt} - \Delta u + \int_0^{\infty} g(s)\Delta u(x,t-s)ds = 0, x \in \Omega, t > 0 \\ u_{tt} = -\left(\frac{\partial u}{\partial \nu}(x,t) - \int_0^{\infty} g(s)\frac{\partial u}{\partial \nu}(x,t-s)ds\right), x \in \Gamma_1, t > 0 \\ u(x,t) = 0, x \in \Gamma_0, t > 0 \\ u(x,-t) = u^0(x,t), x \in \Omega, t > 0 \\ u_t(x,0) = u_1(x), x \in \Omega \\ u(x,0) = u_0(x), x \in \Omega, \end{cases}$$

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- x is the space variable along the beam in the bounded domain.
- t denotes the time variable.
- u denote the transverse displacements of waves.
- u_t represent the damping terms.
- $g(.)$ represent the viscoelastic materials which is a kind of material that has the properties of keeping past information (memories) and which are able to be used in the future.
- $u^0 : \Omega \times (-\infty, 0] \to \mathbb{R}$ is the prescribed past history of u.

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The relaxation function g is differentiable function such that, for $s \geq 0$

- (2) $$g(s) \geq 0, \quad 1 - \int_0^\infty g(s)ds = \ell > 0,$$

- (3) $$\exists\, \zeta_0, \zeta_1 > 0 : \quad -\zeta_1 g(t) \leq g'(t) \leq -\zeta_0 g(t), \quad \forall\, t \in \mathbb{R}.$$

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A natural question arises in the context of the viscoelastic Wave problem with dynamic boundary conditions: Is it possible to stabilize the system by considering some memory damping acting in the first and second equation? The main goal of this talk is exactly to give an answer to this question, namely: to prove that a memory damping is strong enough, via transmission process ($u|\Gamma = \eta$), to assure the asymptotic stability of the whole system. This is a hard problem since we have dynamic boundary conditions, instead of static one. In other words: we have additional terms in the energy which come from a dynamic boundary so that we have to deal with these terms when we estimate the integral inequalities of energy.

In order to prove the existence of solutions of problem (1), we follow the approach of Dafermos [1], by considering a new auxiliary variable as the relative displacement history of u as follows:

$$\eta := \eta^t(x,s) = u(x,t) - u(x,t-s) \quad \text{in} \quad \Omega \times (0,\infty) \times (0,\infty).$$

and the weighted L^2- spaces

$$\begin{aligned} \mathcal{M} &= L^2_g(\mathbb{R}_+; H^1_{\Gamma_0}(\Omega)) \\ &= \left\{ \xi : \mathbb{R}_+ \to H^1_{\Gamma_0}(\Omega)) : \int_0^\infty g(s) \|\nabla \xi(s)\|_2^2 ds < \infty \right\}, \end{aligned}$$

which is a Hilbert space endowed with inner product and norm consecutively

$$\langle \xi, \zeta \rangle_{\mathcal{M}} = \int_0^\infty g(s) \left(\int_\Omega \nabla \xi(s) \nabla \zeta(s) dx \right) ds,$$

and

$$\|\xi\|_{\mathcal{M}}^2 = \int_0^\infty g(s) \|\nabla \xi(s)\|_2^2 ds.$$

Our analysis is given on the phase space

$$\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times \mathcal{M}. \quad (4)$$

If we denote $V := (u, u_t, \gamma_1(u_t), \eta)$, clearly, $\mathcal{H}$ is a Hilbert space with respect to the inner product

$$
\begin{aligned}
\langle V_1, V_2\rangle_{\mathcal{H}} &= (1-g_0)\int_\Omega \nabla u_1.\nabla u_2 dx + \int_\Omega \upsilon_1.\upsilon_2 dx \\
&+ \int_{\Gamma_1} w_1.w_2 d\sigma \\
&+ \int_0^\infty g(s)\left(\int_\Omega \nabla\eta_1(s).\nabla\eta_2(s)dx\right) ds, \qquad (5)
\end{aligned}
$$

with $V_1 = (u_1, \upsilon_1, w_1, \eta_1)^{\mathsf{T}}$ and $V_2 = (u_2, \upsilon_2, w_2, \eta_2)^{\mathsf{T}}$.

Therefore, problem (1) is equivalent to
(6)
$$
\begin{cases}
u_{tt} - \ell\Delta u - \int_0^\infty g(s)\Delta\eta^t(x,s)ds = 0, x \in \Omega, t > 0,\\
u_{tt} = -\left(\frac{\partial u}{\partial \nu}(x,t) + \int_0^\infty g(s)\frac{\partial u}{\partial \nu}(x,t-s)ds\right), x \in \Gamma_1, t > 0,\\
\eta_t^t(x,t) + \eta_s^t(x,s) = u_t(x,t), x \in \Omega, t > 0, s > 0,\\
u(x,t) = \eta^t(x,0) = 0, x \in \Gamma_0,\ \ t > 0,\\
u(x,-t) = u^0(x,t), x \in \Omega,\ \ t > 0,\\
u_t(x,0) = u_1(x), x \in \Omega,\\
u(x,0) = u_0(x), x \in \Omega.
\end{cases}
$$

the problem (6) is formally equivalent to the following abstract evolution equation in the Hilbert space $\mathcal{H}$

$$(7) \qquad \begin{cases} V'(t) = \mathcal{A}V(t), \quad t > 0 \\ V(0) = V_0, . \end{cases}$$

such that $V_0 = (u_0, u_1, \gamma_1(u_1), \eta^0)^{\mathsf{T}}$ and the operator $\mathcal{A}$ is defined by

$$(8)\quad \mathcal{A}\begin{pmatrix} u \\ \upsilon \\ \omega \\ \eta \end{pmatrix} = \begin{pmatrix} \upsilon \\ (1-g_0)\Delta u + \int_0^\infty g(s)\Delta\eta(s)ds \\ -\frac{\partial u}{\partial \nu} - \int_0^\infty g(s)\frac{\partial \omega}{\partial \nu}(x,t-s)ds \\ -\frac{\partial \eta}{\partial s} + \upsilon \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u,\upsilon,\omega,\eta) \in \left(H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) \times \\ L^2(\Gamma_1)\times \mathcal{M},\ \text{s.t} \\ \\ (1-g_0)u + \int_0^\infty g(s)\eta(s)ds \in L^2(\Omega), \\ \\ \omega = \gamma_1(u) = u_0(.,0), \eta(0) = 0 \text{ on } \Gamma_1 \end{array} \right\}$$

Theorem

Let $V_0 \in \mathcal{H}$. Then, system (6) has a unique weak solution

$$V \in \mathcal{C}(\mathbb{R}^+; \mathcal{H})$$

Moreover, if $V_0 \in D(\mathcal{A})$, then the solution of (7) satisfies

$$V \in \mathcal{C}^1(\mathbb{R}^+; \mathcal{H}) \cap \mathcal{C}(\mathbb{R}^+; D(\mathcal{A}))$$

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Proof:

For $V = (u, \upsilon, \omega, \eta)^{\mathsf{T}} \in D(\mathcal{A})$, we have

$$\langle \mathcal{A}V, V\rangle_{\mathcal{H}} =$$

$$\left\langle \begin{pmatrix} \upsilon \\ (1-g_0)\Delta u + \int_0^\infty g(s)\Delta\eta(s)ds \\ -\frac{\partial u}{\partial \nu} - \int_0^\infty g(s)\frac{\partial u}{\partial \nu}(x,t-s)ds \\ -\frac{\partial \eta}{\partial s} + \upsilon \end{pmatrix}, \begin{pmatrix} u \\ \upsilon \\ \omega \\ \eta \end{pmatrix} \right\rangle$$

(9)

$$
\begin{aligned}
\langle \mathcal{A}V, V\rangle_{\mathcal{H}} &= (1-g_0)\int_\Omega \nabla \upsilon.\nabla u dx + (1-g_0)\int_\Omega \Delta u.\upsilon dx \\
&+ (1-g_0)\int_\Omega \int_0^\infty g(s)\Delta\eta(s)\upsilon(s)ds \\
&+ \int_{\Gamma_1}\left(\frac{-\partial u}{\partial \nu} - \int_0^\infty g(s)\frac{\partial u}{\partial \nu}(x,t-s)ds\right)\omega d\sigma \\
&+ \left\langle \frac{-\partial \eta}{\partial s} + \upsilon, \eta \right\rangle_{L^2_g}.
\end{aligned}
$$

Noting that

$$(10) \quad \int_{\Gamma_1} \left(\frac{-\partial u}{\partial \nu} - \int_0^\infty g(s) \frac{\partial u}{\partial \nu}(x, t-s) ds \right) \omega d\sigma = 0$$

By exploiting Green's formula, integrating by parts and using the fact that $\eta(0) = 0$ (from the definition of $D(\mathcal{A})$), we obtain

$$\left\langle \frac{-\partial \eta}{\partial s}, \eta \right\rangle_{L^2_g} = \frac{1}{2} \int_0^\infty g'(s) \|\nabla \eta(s)\|^2 ds.$$

Inserting the previous identity into (9) , we get

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} = \frac{1}{2} \int_0^\infty g'(s) \|\nabla \eta(s)\|^2 ds$$

which implies that

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} \leq 0.$$

Next, we shall prove that $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. Indeed, let $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, and we look for $W = (\omega_1, \omega_2, \omega_3, \omega_4)^T \in D(\mathcal{A})$ satisfying

$$(\lambda I - \mathcal{A})W = F \tag{11}$$

which is equivalent to

(12)

$$\begin{cases} \lambda\omega_1 - \omega_2 = f_1 \\ -(1-g_0)\Delta\omega_1 + \lambda\omega_2 - \int_0^\infty g(s)\Delta\omega_4(s)ds = f_2 \\ \lambda\omega_3 + \frac{\partial\omega_1}{\partial\nu} + \int_0^\infty g(s)\frac{\partial\omega_3(s)}{\partial\nu}ds = f_3 \\ -\omega_2 + \lambda\omega_4 + \frac{\partial}{\partial s}\omega_4 = f_4. \end{cases}$$

the first equation in (12) gives

$$\omega_2 = \lambda\omega_1 - f_1$$

and the last equation in (12) with $\eta(0) = 0$ has unique solution

$$\omega_4(s) = \left(\int_0^s e^y (f_4(y) + \omega_2(y))dy\right) e^{-s}.$$

From the first and the second equation in (12) we can deduce the following

$$(13)\quad \lambda^2\omega_1 - (1-g_0)\Delta\omega_1 = (f_2 + \lambda f_1) + \int_0^\infty g(s)\omega_4(s)ds.$$

Putting $\bar{u} = \omega_1 + \int_0^\infty g(s)\omega_3(s)ds$. Then from equation (13), $\bar{u}$ must satisfy

(14)

$$\lambda^2 \bar{u} - (1-g_0)\Delta \bar{u} = \lambda^2 \int_0^\infty g(s)\omega_3(s)ds + (f_2 + \lambda f_1) - (1-g_0)\int_0^\infty g(s)\Delta\omega_3(s)ds + \int_0^\infty g(s)\omega_4(s)ds$$

with the boundary conditions

(15) $\bar{u} = 0$ on Γ_0

(16) $\frac{\partial \bar{u}}{\partial \nu} = f_3 - \lambda \bar{u} + \lambda(1-\ell)u_0$ on Γ_1.

The variational formulation of (14) is given by:

$$a(\bar{u}, \varphi) = l(\varphi) \quad \forall\, \varphi \in H^1_{\Gamma_0}(\Omega) \tag{17}$$

where

$$a(\bar{u}, \varphi) = \int_\Omega \left[\lambda^2 \bar{u}.\varphi + (1 - g_0)\, \nabla \bar{u}.\nabla \varphi\right] dx + \lambda \int_{\Gamma_1} \bar{u}(\sigma)\varphi(\sigma) d\sigma$$

and

$$
\begin{aligned}
l(\varphi) = \lambda^2 \int_\Omega \int_0^\infty g(s)\omega_3(s)ds\varphi dx + \int_\Omega (f_2 + \lambda f_1)\varphi dx \\
+ (1 - g_0) \int_\Omega \int_0^\infty g(s)\nabla\omega_3(s)ds\nabla\varphi dx \\
+ \int_\Omega \int_0^\infty g(s)\omega_4(s)ds\varphi dx + \lambda \int_{\Gamma_1} u_0(\sigma)\varphi(\sigma)d\sigma
\end{aligned}
$$

using the Lax-Milgram Theorem, we conclude that (14) has a unique solution $\bar{u}$ in $H^1_{\Gamma_0}(\Omega)$. By classical regularity arguments, we conclude that the solution $\bar{u}$ of (14) belongs into $H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$ and satisfies (14). Consequently, using (18) and (18), we deduce that (6) has a unique solution $V \in D(\mathcal{A})$. This proves that $(\lambda I - \mathcal{A})$ is surjective and hence $\mathcal{A}$ is an infinitesimal generator of a linear $\mathcal{C}_0$ semigroup of contractions on $\mathcal{H}$.

Decreasing Energy: $E(t) \searrow$

The energy associated with (6) is defined by

$$(18) \qquad E(t) = \frac{1}{2}\left\{\|u_t(t)\|_{\Gamma_1}^2 + \|\nabla u(t)\|_2^2 + \|\eta\|_{\mathcal{M}}\right\},$$

Lemma

The functional defined in (18) satisfies the following inequality

$$(19) \qquad E'(t) \leq \frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta(s)\|_2^2 ds, \ \forall\ t \geq 0,$$

Proof:

By multiplying the first equation in (6) by $u_t(t)$, and integrating over Ω we get

$$(20) \qquad \begin{aligned} 0 = & \frac{1}{2}\frac{d}{dt}\left\{\|u_t(t)\|^2_{\Gamma_1} + \|\nabla u(t)\|_2^2\right\} \\ & + \int_0^\infty g(s)\int_\Omega \nabla\eta(s)\nabla u_t(t) ds dx. \end{aligned}$$

Since

$$u_t(x,t) = \eta_t(x,s) + \eta_s(x,s), \quad (x,s) \in \Omega \times \mathbb{R}^+, \ t \geq 0,$$

we have

$$\int_0^\infty g(s)\int_\Omega \nabla\eta(s)\nabla u_t(t)dxds$$
$$= \frac{1}{2}\int_0^\infty g(s)\frac{d}{dt}\|\nabla\eta(s)\|_2^2 ds - \frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta(s)\|_2^2 ds$$
$$+ \int_0^\infty g(s)\int_\Omega \nabla\eta(s)\nabla\eta_t(t)dxds.$$

Due to Young's inequality, we have for any $\delta > 0$

(21)
$$\int_0^\infty g(s)\int_\Omega \nabla\eta(s)\nabla\eta_t(t)dxds$$
$$\leq \int_0^\infty g(s)\left(\frac{1}{4\delta}\|\nabla\eta(s)\|_2^2 + \delta\|\nabla\eta_t\|_2^2\right) ds$$
$$\leq \delta\left(\int_0^\infty g(s)ds\right)\|\nabla\eta_t\|_2^2 + \frac{1}{4\delta}\int_0^\infty g(s)\|\nabla\eta(s)\|_2^2 ds$$

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Exponential Stability Conditions

The necessary and sufficient conditions for the exponential stability of the C_0-semigroup of contractions on a Hilbert space were obtained by Gearhart [2] and Huang [3] independently, see also Prüss [4]. We will use the following result due to Gearhart.

Lemma

A Semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0}$ of contractions on a Hilbert space $\mathcal{X}$ is exponentially stable if and only if

$$i\mathbb{R} \equiv \{i\beta; \quad \beta \in \mathbb{R}\} \subset \rho(\mathcal{A}) \tag{22}$$

and

$$\limsup_{|\beta|\to\infty} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{X}} < \infty \tag{23}$$

Main Result

Theorem

The semigroup of system (6) decays exponentially as

$$\|e^{t\mathcal{A}}V_0\|_{\mathcal{H}} \leq Ce^{-\gamma t}\|V_0\|_{D(\mathcal{A})}, \quad \forall\ V_0 \in D(\mathcal{A}),\ t > 0. \tag{24}$$

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Proof of Theorem

The proof is splinted into two parts: the first part consists to prove (22) which is equivalent to prove the following two assertions

1. If β is a real number, then $(i\beta I - \mathcal{A})$ is injectif and
2. If β is a real number, then $(i\beta I - \mathcal{A})$ is surjectif.

Lemma

If β is a real number, then $i\beta$ is not an eigenvalue of $\mathcal{A}$.

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Proof of Lemma5

We will show that the equation

$$\mathcal{A}Z = i\beta Z \tag{25}$$

with $Z = (u, v, \omega, \eta)^{\mathsf{T}} \in D(\mathcal{A})$ and $\beta \in \mathbb{R}$ has only the trivial solution. Equation (25) can be written as

$$i\beta u - v = 0 \tag{26}$$

$$i\beta v - (1 - g_0)\Delta u - \int_0^\infty g(s)\Delta\eta(s)ds = 0 \tag{27}$$

$$i\beta\omega + \frac{\partial u}{\partial\nu} + \int_0^\infty g(s)\frac{\partial\omega(s)}{\partial\nu}ds = 0 \tag{28}$$

$$i\beta\eta + \frac{\partial\eta}{\partial s} - v = 0 \tag{29}$$

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By taking the inner product of (25) with $Z \in D(\mathcal{A})$ and using (19), we get:

$$
\begin{aligned}
\Re(\langle \mathcal{A}Z, Z\rangle_{\mathcal{H}}) &\leq \int_0^\infty g'(s)\|\nabla\eta(s)\|^2 ds \\
&\leq -\int_0^\infty g(s)\|\nabla\eta(s)\|^2 ds \\
&= -\|\eta\|^2_{\mathcal{M}} \\
&\leq 0
\end{aligned} \tag{30}
$$

Thus we obtain that $\eta = 0$, moreover, as η satisfies (29), by integration we obtain

$$\eta(s) = \left(\int_0^s e^{i\beta y}\upsilon(y))dy\right) e^{-i\beta s}.$$

Since $\eta = 0$ we deduce that $\upsilon = 0$ and from (26) we have $u = 0$. Moreover, as $\omega = \gamma_1(u) = u_0(.,0)$, we obtain also $\omega = 0$. Thus the only solution of (25) is the trivial one. Hence the proof is completed.

- Next, we show that $i\beta I - \mathcal{A}$ is surjective.

In fact, for $F = (f_1, f_2, f_3, f_4)^\mathsf{T} \in \mathcal{H}$, let $V = (u, \upsilon, \omega, \eta)^\mathsf{T} \in D(\mathcal{A})$ solution of

$$(i\beta I - \mathcal{A})V = F \tag{31}$$

which is

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$$(32)\quad \begin{cases} i\beta u - \upsilon = f_1 \\ -(1-g_0)\Delta u + i\beta \upsilon - \int_0^\infty g(s)\Delta\eta(s)ds = f_2 \\ i\beta\omega + \frac{\partial u}{\partial \nu} + \int_0^\infty g(s)\frac{\partial \omega(s)}{\partial \nu}ds = f_3 \\ -\upsilon + i\beta\eta + \frac{\partial \eta}{\partial s} = f_4. \end{cases}$$

The first equation in (32) gives

$$(33)\qquad \upsilon = i\beta\omega_1 - f_1.$$

The last equation in (32) with $\eta(0) = 0$ has unique solution

$$(34)\qquad \omega_4(s) = \left(\int_0^s e^{i\beta y}(f_4(y) + \omega_2(y))dy\right) e^{-i\beta s}$$

Another time, from the first and the second equation in (32) we can deduce the following

(35)
$$(i\beta)^2\omega_1 - (1-g_0)\Delta\omega_1 = (f_2 + i\beta f_1) + \int_0^\infty g(s)\omega_4(s)ds$$

Setting $\bar{u} = \omega_1 + \int_0^\infty g(s)\omega_3(s)ds$. Then $\bar{u}$ verify

(36)
$$(i\beta)^2\bar{u} - (1-g_0)\Delta\bar{u} = (i\beta)^2\int_0^\infty g(s)\omega_3(s)ds + (f_2 + i\beta f_1)$$
$$-(1-g_0)\int_0^\infty g(s)\Delta\omega_3(s)ds + \int_0^\infty g(s)\omega_4(s)ds$$

with the boundary conditions

$$(37) \qquad \bar{u} = 0 \quad \text{on} \quad \Gamma_0$$

$$(38) \qquad \frac{\partial \bar{u}}{\partial \nu} = f_3 - i\beta\bar{u} + i\beta u_0(x)(1-\ell) \quad \text{on} \quad \Gamma_1.$$

The variational formulation of (36) is:

$$(39) \qquad b(\bar{u}, \varphi) = l(\varphi) \quad \forall\, \varphi \in H^1_{\Gamma_0}(\Omega)$$

where

$$b(\bar{u}, \varphi) = \int_\Omega \left[(i\beta)^2 \bar{u}.\varphi + (1-g_0)\,\nabla\bar{u}.\nabla\varphi\right] dx + i\beta \int_{\Gamma_1} \bar{u}(\sigma)\varphi(\sigma)d\sigma$$

and

$$I(\varphi) = (i\beta)^2 \int_\Omega \int_0^\infty g(s)\omega_3(s)ds\varphi dx + \int_\Omega (f_2 + i\beta f_1)\varphi dx$$
$$+ (1-g_0)\int_\Omega \int_0^\infty g(s)\nabla\omega_3(s)ds\nabla\varphi dx$$
$$+ i\beta \int_{\Gamma_1} u_0(\sigma)\varphi(\sigma)d\sigma$$

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Lemma

The resolvent operator of $\mathcal{A}$ satisfies (23), that's

$$\limsup_{|\beta|\to\infty} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{X}} < \infty.$$

Proof of (23)

Suppose that condition (23) is false. By Banach-steinhaus Theorem, there exists a sequence of real numbers $\beta_n \to +\infty$ and a sequence of vectors

(40) $$Z_n = (u_n, v_n, \omega_n, \eta_n)^{\mathsf{T}} \in D(\mathcal{A}) \quad \text{with } \|Z_n\|_{\mathcal{H}} = 1$$

such that

(41) $$\|(i\beta_n I - \mathcal{A})Z_n\|_{\mathcal{H}} \to 0 \text{ as } n \to \infty.$$

That's

$$(i\beta_n u_n - \upsilon_n) \equiv f_n \to 0, \text{ in } H^1_{\Gamma_0}(\Omega)$$

$$\left(i\beta_n \upsilon_n - (1-g_0)\Delta u_n - \int_0^\infty g(s)\Delta\eta_n(s)ds\right) \equiv g_n \to 0, \text{ in } L^2(\Omega)$$

$$\left(i\beta_n \omega_n + \frac{\partial u_n}{\partial \nu} + \int_0^\infty g(s)\frac{\partial \omega_n(s)}{\partial \nu}ds\right) \equiv h_n \to 0, \text{ in } L^2(\Gamma_1)$$

$$\left(i\beta_n \eta_n + \frac{\partial \eta_n}{\partial s} - \upsilon_n\right) \equiv k_n \to 0, \text{ in } \mathcal{M}.$$

$$(42) \qquad \left|\Re\left\langle (i\beta_n I - \mathcal{A})Z_n, Z_n\right\rangle_{\mathcal{H}}\right| \leq \|(i\beta_n I - \mathcal{A})Z_n\|_{\mathcal{H}}.$$

Using the hypothesis on g , we find that

$$(43) \qquad \eta_n \to 0 \text{ in } L^2_g(\mathbb{R}_+; H^1_{\Gamma_0}(\Omega))$$

and

(44)

$$\eta_n(s) = \left(\int_0^s e^{i\beta y} k_n(y)\right) e^{-i\beta s} + \left(\int_0^s e^{i\beta y} \upsilon_n(y) dy\right) e^{-i\beta s}.$$

By exploiting the convergence (43) and (44), we can deduce from (42) that

$$(45) \qquad \upsilon_n \to 0 \text{ in } L^2(\Omega) \text{ and } u_n \to 0 \text{ in } L^2(\Omega).$$

Now, multiplying the equation (42) by υ_n and (42) by u_n, adding them and taking the real parts , we obtain

(46)

$$\|\upsilon_n\|_2^2+(1-g_0)\|\nabla u_n\|_2^2+\int_0^\infty g(s)\nabla\eta_n(s)\nabla u_n(t)ds \to 0 \text{ in } L^2(\Omega).$$

According to Young's inequality, we have for any $\delta > 0$

$$\int_0^\infty g(s)\int_\Omega \nabla\eta_n(s)\nabla u_n(t)dxds \leq$$
$$\int_0^\infty g(s)\left(\frac{1}{4\delta}\|\nabla\eta_n(s)\|_2^2+\delta\|\nabla u_n\|_2^2\right)ds$$
$$\leq \delta\left(\int_0^\infty g(s)ds\right)\|\nabla u_n\|_2^2+\frac{1}{4\delta}\int_0^\infty g(s)\|\nabla\eta_n(s)\|_2^2 ds$$
$$= \delta g_0\|\nabla u_n\|_2^2+\frac{1}{4\delta}\|\eta\|_{L_g^2}^2.$$

Replacing the last inequality in (46), for δ sufficiently small we get

$$\nabla u_n \to 0 \quad \text{in} \quad L^2(\Omega). \tag{47}$$

Consequently we have

$$u_n \to 0 \quad \text{in} \quad H^1_{\Gamma_0}(\Omega). \tag{48}$$

By using (42) and trace theorem we get

$$\omega_n \to 0 \quad \text{in} \quad L^2(\Gamma_1) \tag{49}$$

which contradicts (40) . Thus (23) is proved.

Open Problem

- The associated Controllability (Exact or approximate) problem. That's $u_{tt} - \Delta u + \int_0^\infty g(s) \Delta u(x, t-s) ds = f$.
- Stability issue for different powers of $-\Delta$ in the damping term:

$$u_{tt} - \Delta u + \int_0^\infty g(s) \Delta^\theta u(x, t-s) ds = 0, \quad 0 \le \theta \le \frac{1}{2}.$$

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Thank you