STABILIZATION OF COUPLED WAVE EQUATIONS WITH BOUNDARY DAMPING

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Plan

1. Motivation
2. Problem
3. Well-posedness and strong stability
4. Exponential stability (Case : $\rho = 1$)
5. Polynomial stability (Case : $\rho \neq 1$)

Motivation

• In 1995, Cox-Zuazua studied the energy of the solution of the wave equation

$$\begin{cases} \rho^2(x)\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0, & \text{in }]0,1[\times]0,+\infty[\\ u(x,0) = u_0(x) \ \ , \ \partial_t u(x,0) = v_0(x), & \forall x \in [0,1] \end{cases}$$

under the damping condition

$$\partial_x u(1,t) + \partial_t u(1,t) = 0 \ \ \forall t \in]0,+\infty[.$$

• If the density ρ is suitably chosen, the dissipation of the energy through the right bound is sufficient to lead to exponential decay of solution.

• In the case of the density equal to one and no damping applied, i.e., when

$$u(1,t) = u(0,t) = 0 \ \ \forall t \in]0,+\infty[,$$

then the energy of solutions is conserved.

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Question : How the stability properties are effected if we couple the exponentially stable wave equations to the conservative one ?

Problem

• The functions u_1 and u_2 denote, respectively, the transverse displacement of a string of unit length in the case the density $\rho^2 \neq 1$ and $\rho = 1$.

$$\begin{cases} \rho^2 \partial_t^2 u_1(x,t) - \partial_x^2 u_1(x,t) + \partial_t u_2(x,t) = 0, & \text{in }]0,1[\times]0,+\infty[\\ \partial_t^2 u_2(x,t) - \partial_x^2 u_2(x,t) - \partial_t u_1(x,t) = 0, & \text{in }]0,1[\times]0,+\infty[\\ u_1(0,t) = 0 & \forall t \in]0,+\infty[\\ u_2(0,t) = u_2(1,t) = 0 & \forall t \in]0,+\infty[\end{cases} \tag{2.1}$$

with the following initial conditions

$$u_1(x,0) = u_1^0(x),\ \partial_t u_1(x,0) = u_1^1(x),\ u_2(x,0) = u_2^0(x),\ \partial_t u_2(x,0) = u_2^1(x)$$

and the boundary dissipation law

$$\partial_x u_1(1,t) + \partial_t u_1(1,t) = 0 \quad \forall t \in]0,+\infty[. \tag{2.2}$$

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- Let us set

$$H_L^1(0,1) = \{y \in H^1(0,1) : y(0) = 0\}.$$

- Define the energy space H

$$H = H_L^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1)$$

equipped with the inner product defined by

$$(U, U_1)_H := \int_0^1 (u'\bar{u}_1' + \rho^2 v\bar{v}_1 + y'\bar{y}_1' + z\bar{z}_1)dx,$$

$$\forall U = (u, v, y, z), U_1 = (u_1, v_1, y_1, z_1) \in H.$$

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• Let (u_1, u_2) be a regular solution of system (2.1)-(2.2), its associated total energy is defined by

$$E(t) = \frac{1}{2}\int_0^1 \left(|\partial_x u_1(x,t)|^2 + \rho^2|\partial_t u_1(x,t)|^2 + |\partial_x u_2(x,t)|^2 + |\partial_t u_2(x,t)|^2\right) dx \quad \forall t > 0. \tag{2.3}$$

• We have the dissipation law

$$\frac{dE(t)}{dt} = -\left|\partial_x u_1(1,t)\right|^2.$$

Hence, system (2.1)-(2.2) is dissipative in the sense that its associated energy is non increasing with respect to time.

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Question 1 : Does the energy decreases to zero ?
Question 2 : When the energy approach to zero, how fast its decay is, and under what conditions ?

Well-posedness

• Let $U = (u_1, \partial_t u_1, u_2, \partial_t u_2)^t$. Then the system (2.1)-(2.2) can be rewritten as an evolutionary equation in H

$$\begin{cases} \partial_t U = AU(t), \quad t > 0 \\ U(0) = U_0 \end{cases} \tag{3.4}$$

where $U_0 = (u_1^0, u_1^1, u_2^0, u_2^1)^t \in H$ is given.

• $A : D(A) \longrightarrow H$ is defined by :

$$D(A) = \{(u, v, y, z) \in (H^2(0,1) \cap H_L^1(0,1)) \times H_L^1(0,1) \times (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1$$

$$u'(1) + v(1) = 0\}$$

with

$$A\,(u, v, y, z)^t = \left(v, \frac{1}{\rho^2}\Delta u - \frac{1}{\rho^2} z, z, \Delta y + v\right)^t, \ \forall\ U = (u, v, y, z)^t \in D(A).$$

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Proposition

The operator A is m-dissipative in the energy space H. In addition, the linear bounded operator A^{-1} is compact in H.

- Using Lumer-Phillips Theorem, the operator A generates a C_0-semigroup of contraction $(e^{tA})_{t\geq 0}$ on H.

Theorem (Existence and uniqueness)

(1) If $U_0 \in D(A)$, then system (3.4) has a unique strong solution

$$U \in \mathcal{C}([0, +\infty[, D(A)) \cap \mathcal{C}^1([0, +\infty[, H)$$

(2) If $U_0 \in H$, then system (3.4) has a unique weak solution

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Strong stability

• A generates a contraction semigroup and its resolvent is compact in H, using Arendt-Batty theorem, system (2.1)-(2.2) is strongly stable, i.e,

$$\lim_{t\to+\infty} ||e^{tA}U_0||_X = 0 \quad \forall U_0 \in H$$

if and only if A does not have pure imaginary eigenvalues.

theorem (Strong Stability)

The semigroup of contraction $(e^{tA})_{t\geq 0}$ is strongly stable on the energy space H if and only if the coefficient ρ satisfies the following condition

$$\rho^4 + \left(2 - \frac{(k_1^2+k_2^2)((k_1^2+k_2^2)\pi^2 - 1)}{k_1^2k_2^2\pi^2}\right)\rho^2 + \frac{k_1^2+k_2^2}{k_1^2k_2^2\pi^2} + 1 \neq 0 \ \forall (k_1,k_2) \in K, \tag{3.5}$$

where $K = \{(k_1,k_2) \in (\mathbb{N}^*)^2; k_1 \geq k_2 + 1 \text{or } k_1 \leq k_2 - 1\}$.

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Remark

Let $\mathcal{D}$ be the set of solutions of

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then the condition (3.5) is equivalent to $\rho^2 \notin \mathcal{D}$.
Moreover, $\mathcal{D}$ is a countable closed set which implies that the strong stability is true for almost any value of ρ^2.
In addition,

$$\mathcal{D} \cap \mathbb{Q} = \emptyset. \tag{3.6}$$

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Exponential stability (Case :$\rho = 1$)

Theorem

Assume that $\rho = 1$, then the semigroup e^{tA} is exponentially stable, i.e., there exist a constant $M > 0$, and $\omega > 0$ such that

$$\left\| e^{tA} U_0 \right\| \leq M e^{-\omega t} \left\| U_0 \right\| \quad \forall t > 0, U_0 \in D(A). \tag{4.7}$$

- Using Huang and Prüss, inequality (4.7) holds if and only if the following two conditions are satisfied :

(H1) $i\mathbb{R} \subset \rho(A)$,

(H2) $\limsup_{|\lambda| \to +\infty, \lambda \in \mathbb{R}} \|(i\lambda I - A)^{-1}\| < +\infty.$

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Spectral Theory

Proposition

In the case $\rho = 1$, there exists $N \in \mathbb{N}$ such that

$$\{\mu_k\}_{k^* \in \mathbb{Z}, |k| \geq N} \subset \sigma(A) \tag{4.8}$$

where

$$\mu_k = \frac{1}{2} \ln(\cos(1)) + ik\pi + o(1) \quad , k \in \mathbb{Z}^*, |k| \geq N$$

- Graphical interpretation of the exponential stability of the system.

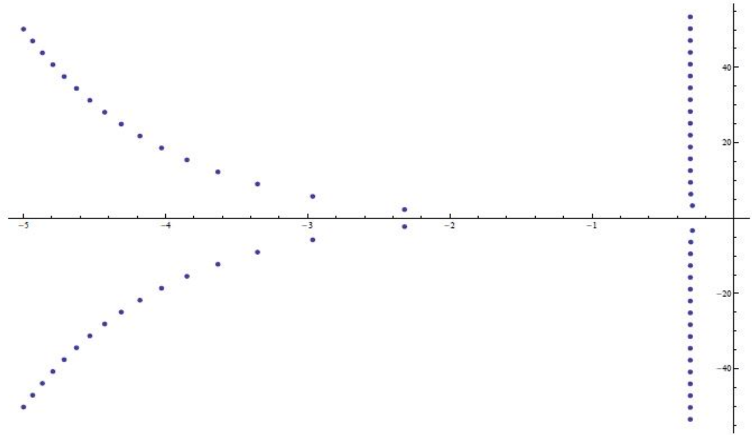

FIGURE: Spectrum of A. Case : $\rho = 1$.

• Let λ be an eigenvalue of A with associated eigenvector $U = (u, v, y, z)$.

•$AU = \lambda U$ is equivalent to

$$\begin{cases} y'''' - 2\lambda^2 y'' + \lambda^2(1+\lambda^2)y = 0 \\ -y''(1) - \frac{1}{\lambda}y'''(1) + \lambda y'(1) = 0 \\ y''(0) = 0 \\ y(1) = y(0) = 0 \end{cases} \tag{4.9}$$

•λ is an eigenvalue of A if and only if (4.9) admits a non trivial solution.

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- The solution of (4.9) is given by :

$$y(x) = \sum_{i=1}^{4} C_i e^{t_i(\lambda)x}$$

where

$$\begin{aligned} t_1(\lambda) &= \sqrt{\lambda^2 + i\lambda}, \\ t_2(\lambda) &= -t_1(\lambda), \\ t_3(\lambda) &= \sqrt{\lambda^2 - i\lambda}, \\ t_4(\lambda) &= -t_3(\lambda). \end{aligned}$$

- The boundary conditions may be written as the following system :

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{t_1} & e^{t_2} & e^{t_3} & e^{t_4} \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ h_\lambda(t_1)e^{t_1} & h_\lambda(t_2)e^{t_2} & h_\lambda(t_3)e^{t_3} & h_\lambda(t_4)e^{t_4} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $h_\lambda(t) = t^2 + \frac{1}{\lambda}t^3 - \lambda t$.

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where $h_\lambda(t) = t^2 + \frac{1}{\lambda}t^3 - \lambda t.$

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$$det(M(\lambda)) = 2\lambda \sinh(t_1) \sinh(t_3) + t_1 \cosh(t_1) \sinh(t_3) + t_3 \cosh(t_3) \sinh(t_1).$$

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Condition $(H2)$:

• Contradiction argument : Suppose that there exists a sequence $(\lambda_n)_n \subset \mathbb{R}$ and a sequence $(U^n)_n = (u^n, v^n, y^n, z^n)_n \subset D(A)$ verifying the following conditions

$$|\lambda_n| \longrightarrow +\infty, \tag{4.11}$$

$$||U^n||_H = 1, \tag{4.12}$$

$$(i\lambda_n I - A)(u^n, v^n, y^n, z^n) = (f_1^n, f_2^n, f_3^n, f_4^n) \longrightarrow 0 \text{ in } H. \tag{4.13}$$

Our goal is to prove that $||U^n||_H = o(1)$. This contradicts equation (4.12) .

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• Since $(U^n)_n$ is uniformly bounded in H, and using the dissipation condition

$$|v^n(1)|^2 = Re\left\langle (i\lambda_n I - A)U^n, U^n \right\rangle_X = o(1). \tag{4.14}$$

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Step 3 : $y'(1) = o(1)$.

- Let $Y = (u, u', y, y')^t$, then (4.13) is equivalent to :

$$Y' = BY + G + \lambda F, \tag{4.18}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\lambda^2 & 0 & i\lambda & 0 \\ 0 & 0 & 0 & 1 \\ -i\lambda & 0 & -\lambda^2 & 0 \end{pmatrix}, F = (F_j) = \begin{pmatrix} 0 \\ -if_1 \\ 0 \\ -if_3 \end{pmatrix}, \tag{4.19}$$

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$$Y(x) = e^{Bx}Y_0 + \int_0^x e^{B(x-z)}G(z)dz + \int_0^x \lambda e^{B(x-z)}F(z)dz \qquad (4.20)$$

where $Y_0 = (u(0), u'(0), y(0), y'(0))^t = (0, u'(0), 0, y'(0))^t$.

- It easy to check that

$$\int_0^x e^{B(x-z)}G(z)dz = o(1).$$

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Polynomial stability (Case : $\rho \neq 1$)

Theorem

Assume that $\rho \neq 1$. If $\rho \in \mathbb{Q}$, then there exists a constant $C > 0$ such that for every initial data $U_0 \in D(A)$, the energy of system (2.1)-(2.2) verifies the following estimate :

$$E(t) \leq C \frac{1}{\sqrt{t}} ||U_0||^2_{D(A)}, \quad \forall t > 0. \tag{5.24}$$

Using Borichev, Tomilov theorem a C_0-semigroup of contractions e^{tA} in a Hilbert space H satisfies (5.24) if and only if

(H1) $i\mathbb{R} \subset \rho(A)$,
and

(H2) $\limsup_{|\lambda| \to +\infty, \lambda \in \mathbb{R}} \frac{1}{|\lambda|^4} ||(i\lambda I - A)^{-1}|| < +\infty.$

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Spectral Theory

Proposition

Assume that $\rho > 1$. There exists $N \in \mathbb{N}^*$ such that

$$\{\lambda_k\}_{k\in\mathbb{Z}^*,|k|\geq N} \cup \{\mu_k\}_{k\in\mathbb{Z},|k|\geq N} \subset \sigma(A) \tag{5.25}$$

where

$$\lambda_k = ik\pi + o(1), k \in \mathbb{Z}^*, |k| \geq N,$$
$$\mu_k = \frac{-1}{2\rho}\ln\left(\frac{\rho+1}{\rho-1}\right) + i\frac{\pi k}{\rho} + o(1), k \in \mathbb{Z}, |k| \geq N$$

Moreover for all $|k| \geq N$, the eigenvalues λ_k and μ_k are simple.

- Graphical interpretation of the polynomial stability of the system

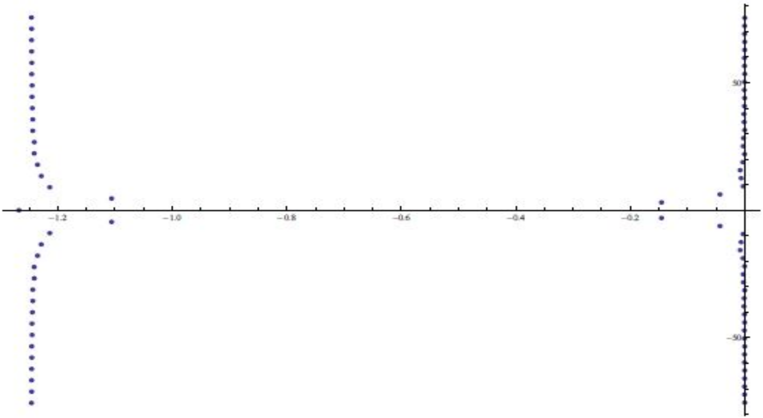

FIGURE: Spectrum of A. Case : $\rho = 2$.

• Let λ be an eigenvalue of A with associated eigenvector $U = (u, v, y, z)$.

•$AU = \lambda U$ is equivalent to

$$\begin{cases} y'''' - \lambda^2(1+\rho^2)y'' + \lambda^2(1+\lambda^2\rho^2)y = 0 \\ -y''(1) - \frac{1}{\lambda}y'''(1) + \lambda y'(1) = 0 \\ y''(0) = 0 \\ y(1) = y(0) = 0. \end{cases} \tag{5.26}$$

•λ is an eigenvalue of A if and only if there is a non trivial solution of (5.26) which satisfies the boundary conditions.

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- The general solution of (5.26) is given by

$$y(x) = \sum_{i=1}^{4} C_i e^{t_i(\lambda)x}$$

where

$$t_1(\lambda) = \sqrt{\frac{\lambda^2(1+\rho^2) + \lambda\sqrt{\lambda^2(1-\rho^2)^2 - 4}}{2}}, \quad t_2(\lambda) = -t_1(\lambda)$$

$$t_3(\lambda) = \sqrt{\frac{\lambda^2(1+\rho^2) - \lambda\sqrt{\lambda^2(1-\rho^2)^2 - 4}}{2}}, \quad t_4(\lambda) = -t_3(\lambda).$$

• The boundary conditions may be written as the following system :

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{t_1} & e^{t_2} & e^{t_3} & e^{t_4} \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ h_\lambda(t_1)e^{t_1} & h_\lambda(t_2)e^{t_2} & h_\lambda(t_3)e^{t_3} & h_\lambda(t_4)e^{t_4} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{t_1} & e^{t_2} & e^{t_3} & e^{t_4} \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ h_\lambda(t_1)e^{t_1} & h_\lambda(t_2)e^{t_2} & h_\lambda(t_3)e^{t_3} & h_\lambda(t_4)e^{t_4} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $h_\lambda(t) = t^2 + \frac{1}{\lambda}t^3 - \lambda t$.
• A non trivial solution y exists if and only if the determinant of $M(\lambda)$ vanishes.
• Let $f(\lambda) = detM(\lambda)$, thus the characteristic equation

$$f(\lambda) = 0.$$

• As we did before, using the asymptotic expansions of t_1 and t_3 and multipling the third and the fourth line of $M(\lambda)$ by $\frac{1}{\lambda^2,}$ we obtain the following asymptotic development

$$f(\lambda) = f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^2} + \frac{f_2(\lambda)}{\lambda^4} + O(\frac{1}{\lambda^4}).$$

• Using the Rouché's theorem, and for λ large enough, the roots of f are close to those of f_0 where

$$f_0(\lambda) = (\rho - 1)^2(\rho + 1)^2 e^{t_1 + t_3}(1 - e^{-2t_3})(\rho + 1 + (\rho - 1)e^{-2t_1}).$$

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•f_0 has two families of roots that we denote λ_k^0 and μ_k^0.

First case :

$$e^{-2t_3} = 1 \iff 2t_3 = 2ik\pi, \quad k \in \mathbb{Z}.$$

Since $t_3 \underset{\infty}{\sim} \lambda$, we get the first family of roots of f_0 is

$$\lambda_k^0 \sim ik\pi + o(1), \quad k \in \mathbb{Z} \text{ large enough.}$$

Second case : $e^{-2t_1} = \frac{\rho+1}{\rho-1}$, then

$$-2t_1 = \ln\left(\frac{\rho+1}{\rho-1}\right) + ik\pi, \quad k \in \mathbb{Z}.$$

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$$\mu_k^0 = -\frac{1}{2\rho}\ln\left(\frac{\rho+1}{\rho-1}\right) + i\frac{k\pi}{\rho} + o(1), \quad k \in \mathbb{Z}.$$

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Conclusion :

• There exists a sequence $(\lambda_n)_n$ of roots of f such that

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Condition $(H2)$

• Using again an argument of contradiction : there exists a sequence $(\lambda_n)_n \subset \mathbb{R}$ and a sequence $(U^n)_n = (u^n, v^n, y^n, z^n)_n \subset D(A)$ verifying the following conditions

$$|\lambda_n| \longrightarrow +\infty, \tag{5.27}$$

$$||U^n||_H = 1, \tag{5.28}$$

$$\lambda_n^4(i\lambda_n I - A)(u^n, v^n, y^n, z^n) = (f_1^n, f_2^n, f_3^n, f_4^n) \longrightarrow 0 \text{ in } H. \tag{5.29}$$

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- We follow the same steps as before, it consists to prove that

$$y_x(1) = o(1). \tag{5.30}$$

- To simplify the computation we translate the problem and we prove that

$$y_x(0) = o(1).$$

- If this is not true, then there exists a constant c such that

$$y_x^n(0) \geq c, \quad \forall n \in \mathbb{N}.$$

- (5.29) can be written as

$$U' = BU + F$$

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$$\begin{cases} u = \sin(\lambda - \frac{1}{2\lambda(\rho^2-1)})y'(0) + \frac{o(1)}{\lambda^3}, \\ y = \sin(\lambda\rho - \frac{1}{2\lambda\rho(\rho^2-1)})y'(0) + \frac{o(1)}{\lambda^4}. \end{cases} \quad (5.31)$$

- Applying the boundary conditions $u(1) = y(1) = 0$, and using the fact that $y'(0) \geq c$, it follows that there exist $m, k \in \mathbb{Z}$ with the same parity such that

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$$\rho^2 m^2 - k^2 = -\frac{\rho^3+1}{\pi\rho(\rho^2-1)} + o(1).$$

- Let $c = -\frac{\rho^3+1}{\pi\rho(\rho^2-1)}$.
- Since we have assumed that $\rho = \frac{p}{q}$ for some $(p, q) \in \mathbb{N}^*$, we deduce

$$\frac{pm - kq}{q} = \frac{c}{pm + kq} + \frac{o(1)}{pm + kq}.$$

$i)$ If $pm - qk = 0$ for an infinity number of pairs (m, k), then $c = o(1)$ and this a contradiction.
$ii)$ Else $pm - kn \neq 0$ for λ large enough and then

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