Existence, uniqueness and global behavior of the solutions to some nonlinear vector equations in a finite dimensional Hilbert space.

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Introduction

Let H be a finite dimensional real Hilbert space, with norm denoted by $\|.\|$. We consider the following nonlinear equation

$$\left(\|u'\|^l u'\right)' + \|A^{\frac{1}{2}}u\|^\beta Au + g(u') = 0, \tag{1}$$

where l and β are positive constants, and A is a positive and symmetric linear operator on H. We denote by $(\cdot,\cdot)$ the inner product in H. The operator A is coercive, which means :

$$\exists \lambda > 0, \ \ \forall u \in D(A), \ \ \ (Au, u) \geq \lambda \|u\|^2.$$

We also define

$$\forall u \in H, \ \ \|A^{\frac{1}{2}}u\| := \|u\|_{D(A^{\frac{1}{2}})},$$

a norm equivalent to the norm in H. We assume that $g : H \to H$ is locally Lipschitz continuous.

Abdelli and Haraux studied the scalar second order ODE

$$\left(|u'|^l u'\right)' + c|u'|^\alpha u' + d|u|^\beta u = 0,$$

they proved the existence and uniqueness of a global solution with initial data $(u_0, u_1) \in \mathbb{R}^2$. They used some modified energy function to estimate the rate of decay and they used the method introduced by Haraux in [2] to study the oscillatory or non-oscillatory of non-trivial solutions. If $\alpha > \frac{\beta(l+1)+l}{\beta+2}$ all non-trivial solutions are oscillatory and if $\alpha < \frac{\beta(l+1)+l}{\beta+2}$ they are non-oscillatory.

M. Abdelli and A. Haraux,
Global behavior of the solutions to a class of nonlinear second order ODE's,
Nonlinear Analysis. **96** (2014), 18-73.

in the case $l = 0$,

A. Haraux
Sharp decay estimates of the solutions to a class of nonlinear second order ODE's
Analysis and Applications. **9** (2011), 49-69.

We consider a degenerate Kirchhoff equation wave equation with a weak frictional damping,

$$\begin{cases} (|u_t|^{l-2}u_t)_t - \left(\int_\Omega |\nabla u|^2 \, dx \right)^\gamma \Delta u + \alpha(t) g(u_t) = 0 \text{ in } \Omega \times (0, +\infty), \\ u = 0 \text{ on } \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \text{ on } \Omega, \end{cases}$$

where $l \geq 2, \gamma \geq 0,$ Ω is a bounded domain in $\mathbb{R}^n$

M. Abdelli and A. Benaissa,
Energy decay of solutions of a degenerate Kirchhoff equation with a weak nonlinear dissipation,
Nonlinear Analysis. **69** (2008), 1999-2008.

In this article, we use some techniques from Abdelli and Haraux to establish a global existence and uniqueness result of the solutions, and under some additional conditions on g (typically $g(s) \sim c\|s\|^{\alpha}s$), we study the asymptotic behavior as $t \to \infty$.

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Global existence and energy conservation for equation

$g : H \to H$ is a locally Lipschitz continuous function which satisfies the following hypothesis :

$$\exists k_1 > 0, \quad k_2 > 0, \quad \forall v, \quad (g(v), v) \geq -k_1 - k_2 \|v\|^{l+2}. \tag{2}$$

Existence of solutions

Theorem 1

Let $(u_0, u_1) \in H \times H$. The problem (1) has a global solution satisfying

$$u \in \mathcal{C}^1(\mathbb{R}^+, H), \quad \|u'\|^l u' \in \mathcal{C}^1(\mathbb{R}^+, H) \quad \text{and} \quad u(0) = u_0, \quad u'(0) = u_1.$$

Proof.

To show the existence of the solution for (1), we consider the auxiliary problem

$$\begin{cases} (\varepsilon + \|u'_\varepsilon\|^2)^{l/2} u''_\varepsilon + l(u'_\varepsilon, u''_\varepsilon)(\varepsilon + \|u'_\varepsilon\|^2)^{l/2-1} u'_\varepsilon + \|A^{\frac{1}{2}} u_\varepsilon\|^\beta A u_\varepsilon + g(u'_\varepsilon) = 0, \\ u_\varepsilon(0) = u_0, \quad u'_\varepsilon(0) = u_1. \end{cases} \tag{3}$$

Here, $\varepsilon > 0$ is a small parameter, devoted to tend to zero.

Then (3), can be rewritten as

$$\begin{cases} u_\varepsilon'' = F_\varepsilon(u_\varepsilon, u_\varepsilon'), \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1. \end{cases} \tag{4}$$

Next, we introduce

$$F_\varepsilon(u_\varepsilon, u_\varepsilon') = l\frac{\|A^{\frac{1}{2}}u_\varepsilon\|^\beta (Au_\varepsilon, u_\varepsilon') + (g(u_\varepsilon'), u_\varepsilon')}{(\varepsilon + \|u_\varepsilon'\|^2)^{l/2}(\varepsilon + (l+1)\|u_\varepsilon'\|^2)} u_\varepsilon' - \frac{\|A^{\frac{1}{2}}u_\varepsilon\|^\beta Au_\varepsilon + g(u_\varepsilon')}{(\varepsilon + \|u_\varepsilon'\|^2)^{l/2}}.$$

i) A priori estimates :
Since the vector field F_ε is locally Lipschitz continuous, the existence and uniqueness of u_ε in the class $\mathcal{C}^2([0,T),H)$, for some $T>0$ is classical. Multiplying (3) by u'_ε, we obtain by a simple calculation

$$\forall t \in [0,T), \quad \|u_\varepsilon(t)\| \leq M_1, \; \|u'_\varepsilon(t)\| \leq M_2, \tag{5}$$

for some constants M_1, M_2 independent of ε. Hence, u_ε and u'_ε are uniformly bounded and u_ε is a global solution, in particular $T>0$ can be taken arbitrarily large.

$$\|((\varepsilon + \|u'_\varepsilon\|^2)^{l/2} u'_\varepsilon)'\| \leq M_3.$$

ii) Passage to the limit :
As a consequence of Ascoli's theorem and a priori estimate (5) combined with (4), we may extract a subsequence which is still denoted for simplicity by (u_ε) such that for every $T > 0$

$$u_\varepsilon \to u \quad \text{in } \mathcal{C}^1((0,T),H),$$

as ε tends to 0. Integrating (3) over $(0,t)$, we get

$$\begin{aligned}(\varepsilon+\|u_\varepsilon'\|^2)^{l/2}u_\varepsilon'(t) \quad &- \quad (\varepsilon+\|u_\varepsilon'\|^2)^{l/2}u_\varepsilon'(t)(0) \\ &= \quad -\int_0^t \|A^{\frac{1}{2}}u_\varepsilon(s)\|^\beta Au_\varepsilon(s)\,ds - \int_0^t g(u_\varepsilon'(s))\,ds.\end{aligned} \tag{6}$$

From (6), we then have, as ε tends to 0

$$(\varepsilon+\|u_\varepsilon'\|^2)^{l/2}u_\varepsilon'(t) \;\rightarrow$$
$$-\int_0^t \|A^{\frac{1}{2}}u(s)\|^\beta Au(s)\,ds - \int_0^t g(u'(s))\,ds + \|u'(0)\|^l u'(0) \;\text{ in }\; \mathcal{C}^0((0,T),H)$$

Hence

$$\|u'\|^l u' = -\int_0^t \|A^{\frac{1}{2}}u(s)\|^\beta Au(s)\,ds - \int_0^t g(u'(s))\,ds + \|u'(0)\|^l u'(0), \tag{7}$$

and $\|u'\|^l u' \in \mathcal{C}^1((0,T),H)$. Finally by differentiating (7) we conclude that u is a solution of (1).

The total energy of the solution u given by the formula

$$E(t) = \frac{l+1}{l+2}\|u'(t)\|^{l+2} + \frac{1}{\beta+2}\|A^{\frac{1}{2}}u(t)\|^{\beta+2}. \tag{8}$$

Remark

It is not difficult to see that the solution u constructed in the existence theorem satisfies the energy identity $\frac{d}{dt}E(t) = -(g(u'(t)), u'(t))$. The following stronger result shows that this identity is true for any solution even if uniqueness is not known. For infinite dimensional equations such as the Kirchhoff equation, both uniqueness and the energy identity for general weak solutions are old open problems.

Theorem 2

Let $(u_0, u_1) \in H \times H$. Then any solution u of (1) such that

$$u \in \mathcal{C}^1(\mathbb{R}^+, H), \quad \|u'\|^l u' \in \mathcal{C}^1(\mathbb{R}^+, H) \quad \text{and} \quad u(0) = u_0, \quad u'(0) = u_1,$$

satisfies the formula

$$\frac{d}{dt}E(t) = -(g(u'(t)), u'(t)). \tag{9}$$

with

$$E(t) = \frac{l+1}{l+2}\|u'(t)\|^{l+2} + \frac{1}{\beta+2}\|A^{\frac{1}{2}}u(t)\|^{\beta+2}. \tag{10}$$

Uniqueness of solution for (u_0, u_1) given

we suppose that

$$\forall R \in \mathbb{R}^+, \exists k_3(R) > 0, \forall (u,v) \in B_R \times B_R, \quad \|g(u) - g(v)\| \leq k_3(R)\|u - v\|, \tag{11}$$

and for some $\alpha \geq l$,

$$\exists \eta_1 > 0, \quad \forall v \in H, \ \|g(v)\| \leq \eta_1 \|v\|^{\alpha+1}. \tag{12}$$

Under these conditions we obtain the following bilateral (forward and backward) uniqueness result.

Proposition 1

For any interval J and any $t_0 \in J$ if a solution u of (1) satisfies

$$u \in \mathcal{C}^1(J,H), \quad \|u'\|^l u' \in \mathcal{C}^1(J,H) \quad \text{and} \quad u(t_0) = u'(t_0) = 0,$$

then $u \equiv 0$.

Proposition 2

Let $(u_0, u_1) \in H \times H$, J an interval of $\mathbb{R}$ and $t_0 \in J$. Then (1) has at most one solution

$$u \in \mathcal{C}^1(J,H), \quad \|u'\|^l u' \in \mathcal{C}^1(J,H) \quad \text{and} \quad u(t_0) = u_0, \quad u'(t_0) = u_1.$$

Proposition 3

Let $a \neq 0$ and $(u_0, u_1) \in H \times H$, J an interval of $\mathbb{R}$ and $t_0 \in J$. Then (1) has at most one solution

$$u \in \mathcal{C}^1(J,H), \quad \|u'\|^l u' \in \mathcal{C}^1(J,H) \quad \text{and} \quad u(t_0) = a, \quad u'(t_0) = 0.$$

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Energy estimates for equation

we suppose that

$$\exists \eta_2 > 0, \quad \forall v, \ \ (g(v), v) \geq \eta_2 \|v\|^{\alpha+2}, \tag{13}$$

for some $\alpha > 0$.

Theorem 5

Assuming $\alpha > l$, there exists a positive constant η such that if u is any solution of (1) with $E(0) \neq 0$

$$\liminf_{t \to +\infty} t^{\frac{l+2}{\alpha - l}} E(t) \geq \eta. \quad (14)$$

Moreover,

(i) if $\alpha \geq \frac{\beta(1+l)+l}{\beta+2}$, then there is a constant $C(E(0))$ depending on $E(0)$ such that

$$\forall t \geq 1, \quad E(t) \leq C(E(0)) t^{-\frac{l+2}{\alpha - l}},$$

(ii) if $\alpha < \frac{\beta(1+l)+l}{\beta+2}$, then there is a constant $C(E(0))$ depending on $E(0)$ such that

$$\forall t \geq 1, \quad E(t) \leq C(E(0)) t^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}.$$

Proof of the upper estimate.

we consider the perturbed energy function

$$E_\varepsilon(t) = E(t) + \varepsilon(\|u\|^{2\gamma} u, \|u'\|^{l} u'), \tag{15}$$

where $l > 0$, $\gamma > 0$ and $\varepsilon > 0$.
We obtain as a consequence of Young's inequality

$$\forall t \geq 0, \quad \frac{1}{2} E(t) \leq E_\varepsilon(t) \leq 2E(t).$$

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Existence of slow and fast solutions for equation

Proposition

Let $\alpha < \frac{\beta(1+l)+l}{\beta+2}$ *and* $c > 0$. *Then the equation*

$$(\|u'\|^l u')' + \|A^{\frac{1}{2}}u\|^\beta Au + c\|u'\|^\alpha u' = 0$$

has an infinity of "fast solutions" with energy comparable to $t^{-\frac{l+2}{\alpha-l}}$ *and an infinity of "slow solutions" with energy comparable to* $t^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}$ *as* t *tends to infinity.*

Proof.

Let $\lambda > 0$ be any eigenvalue of A and $A\varphi = \lambda\varphi$ with $\|\varphi\| = 1$. Let $(v_0, v_1) \in \mathbb{R}^2$ and v be the solution of

$$(|v'|^l v')' + C_1|v|^\beta v + C_2|v'|^\alpha v' = 0, \tag{16}$$

where C_1, C_2 are positive constants to be chosen later. Then $u(t) = v(t)\varphi$ satisfies

$$(\|u'\|^l u')' + \|A^{\frac{1}{2}}u\|^\beta Au + c\|u'\|^\alpha u' = [(|v'|^l v')' + \lambda^{\frac{\beta}{2}+1}|v|^\beta v + c|v'|^\alpha v']\varphi.$$

Choosing $C_1 = \lambda^{\frac{\beta}{2}+1}$ and $C_2 = c$, $u(t) = v(t)\varphi$ becomes a solution of the vector equation.

The existence of an infinity of “fast solutions” and an infinity of “slow solutions” are then an immediate consequence of the same result for the scalar equation proven in

M. Abdelli and A. Haraux,
Global behavior of the solutions to a class of nonlinear second order ODE’s,
Nonlinear Analysis. **96** (2014), 18-73.

Theorem 6

Assuming $l < \alpha < \frac{\beta(l+1)+l}{\beta+2}$, any solution v of (16) satisfies the following alternative : either there is a positive constant C such that

$$\forall t \geq 1, \quad E(t) \leq C(E(0))t^{-\frac{l+2}{\alpha-l}}, \tag{17}$$

or we have

$$\limsup_{t\to\infty} t^{\frac{(\alpha+1)(\beta+2)}{\beta-\alpha)}} E(t) > 0. \tag{18}$$

In the first case, the solution is called fast, and in the second case slow.

Remark

Assume $l < \alpha < \frac{\beta(l+1)+l}{\beta+2}$. Then if a solution v of (16) satisfies (17) there exists a positive constant C such that

$$\forall t \geq 1, \ \ |v(t)| \leq Ct^{-\frac{l+1-\alpha}{\alpha-l}},$$

and

$$\forall t \geq 1, \ \ |v'(t)| \leq Ct^{-\frac{1}{\alpha-l}}.$$

Theorem 8

Let $\alpha < \frac{\beta(l+1)+l}{\beta+2}, \ c > 0, \ d > 0$. Then (16) has an open set of initial data leading to a slow solution, which means a solution satisfying (18).

Proof.

For any solution v of the equation we introduce the new coordinates (z, w) defined by

$$z = \sqrt{\frac{d(l+2)}{(\beta+2)(l+1)}}|v|^{\frac{\beta}{2}}v, \quad w = |v'|^{\frac{l}{2}}v',$$

and since

$$z' = a|z|^{\frac{\beta}{\beta+2}} w^{\frac{2}{l+2}}, \tag{19}$$

with $a = \frac{d(l+2)}{2(l+1)} \left(\frac{(\beta+2)(l+1)}{d(l+2)} \right)^{\frac{\beta+1}{\beta+2}} > 0$ and

$$w' = -a|w|^{-\frac{l}{l+2}} |z|^{\frac{\beta}{\beta+2}} z - c\frac{l+2}{2(l+1)} |w|^{\frac{2\alpha-l+2}{l+2}} \operatorname{sgn}(w) \tag{20}$$

valid whenever $w \neq 0$. For $v < 0,\ v' > 0$, we consider the region $S_{\varepsilon,M}$

$$S_{\varepsilon,M} = \left\{ (z,w) \in \mathbb{R}^2 / z < 0,\ z^2 + w^2 < \varepsilon^2,\ 0 < \frac{w}{|z|} < M \right\}.$$

For any finite M given in advance, we shall show that for ε small enough, the region $S_{\varepsilon,M}$ is positively invariant. To this end we introduce the vector

$$F(z,w) := \left(a|z|^{\frac{\beta}{\beta+2}} w^{\frac{2}{l+2}}, -a|w|^{-\frac{l}{l+2}} |z|^{\frac{\beta}{\beta+2}} z - c\frac{l+2}{2(l+1)} |w|^{\frac{2\alpha-l+2}{l+2}} \mathrm{sgn}(w) \right)$$

so that as long (z,w) remains in $S_{\varepsilon,M}$ we have the equation

$$(z', w') = F(z,w)$$

Setting $B_\varepsilon = \{(z,w) \in \mathbb{R}^2 / z^2 + w^2 \leq \varepsilon^2\}$, since

$$\langle F(z,w),(z,w)\rangle = -c\frac{l+2}{2(l+1)}|w|^{\frac{2(\alpha+2)}{l+2}} \leq 0,$$

we find that the solution cannot escape $S_{\varepsilon,M}$ at a point of ∂B_ε. By backward uniqueness it is clear that (z,w) cannot leave $S_{\varepsilon,M}$ through $(0,0)$. We now show that if ε is small enough, the solution cannot escape at any point of

$$\triangle_M = \{(-\lambda, M\lambda),\ \lambda \in (0,+\infty)\}$$

lying in the closure of B_ε.

Indeed we have

$$F(-\lambda, M\lambda) = \left(aM^{\frac{2}{l+2}}\lambda^{\frac{\beta}{\beta+2}+\frac{2}{l+2}}, aM^{\frac{-l}{l+2}}\lambda^{\frac{\beta}{\beta+2}+\frac{2}{l+2}} - cM^{\frac{2\alpha-l+2}{l+2}}\frac{l+2}{2(l+1)}\lambda^{\frac{2\alpha-l}{l+2}+\frac{2}{l+2}}\right)$$

Since $\frac{2\alpha-l}{l+2} < \frac{\beta}{\beta+2}$ as a consequence of $\alpha < \frac{\beta(l+1)+l}{\beta+2}$, for λ small enough the field at any point of $\triangle_M$ points into the region $S_{\varepsilon,M}$. And smallness of λ is a consequence of smallness of ε whenever $(z, w) \in \triangle_M$.

Finally, since $F(-\lambda, w)$ tends to $(0, +\infty)$ as $w \to 0$, the solution cannot escape $S_{\varepsilon,M}$ at a point lying on the horizontal axis.

Finally, for any trajectory of (19) and (20) lying in any region $S_{\varepsilon,M}$, $\frac{w}{|z|} = |\tan\theta|$ is bounded , when for a fast solution $|\tan\theta|$ blows-up at infinity in t. Hence all solutions confined in $S_{\varepsilon,M}$ are slow solutions.

Theorem (Upper decay estimate for weak solutions)

Assume that g satisfies (13) and $l < \alpha < \frac{\beta(1+l)+l}{\beta+2}$. Let $u(t)$ be the unique solution of equation (1) with $(u_0, u_1) \in H \times H$. Then there exists a constant M such that

$$\|u(t)\| \leq \frac{M}{(1+t)^{\frac{\alpha+1}{\beta-\alpha}}} \quad \forall t \geq 0. \tag{21}$$

The following result is the main progress recorded in

M. Ghisi, M. Gobbino and A. Haraux,
Optimal decay estimates for the general solution to a class of semi-linear dissipative hyperbolic equations
Journal of the European Mathematical Society, to appear.

Theorem (Existence of slow solutions)

Assume that g satisfies (13), $l < 1$ and $l < \alpha < \frac{\beta(1+l)+l}{\beta+2}$. Then, there exist a nonempty open set $\mathcal{S} \subset H \times H$ and a constant M such that, for every $(u_0, u_1) \in \mathcal{S}$, the unique global solution of equation (1) with initial data (u_0, u_1) satisfies

$$\|u(t)\| \geq \frac{M}{(1+t)^{\frac{\alpha+1}{\beta-\alpha}}} \quad \forall t \geq 0. \tag{22}$$

Thank you